1970-06

A theory of generalized filters

Lynn, Freddie Leroy, Jr.

Monterey, California ; Naval Postgraduate School

A THEORY OF GENERALIZED FILTERS

Freddie LeRoy Lynn

United States Naval Postgraduate School

DEPARTMENT OF THE NAVY · NAVAL POSTGRADUATE SCHOOL

THESIS

A THEORY OF GENERALIZED FILTERS

by

Freddie LeRoy Lynn, Jr.

June 1970

A Theory of Generalized Filters

by

Freddie LeRoy Lynn, Jr.
Lieutenant (junior grade), United States Navy
B.S., University of South Carolina, 1969

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN MATHEMATICS

from the

NAVAL POSTGRADUATE SCHOOL
June 1970

ABSTRACT

This manuscript is a collection of many of the known results in the theory of generalized filters ($\mathfrak{Z}$ -filters) as well as an extension of some of the work in this area. The relation of $\mathfrak{Z}$-filters to compactifications and real-compactifications is given special attention. Special emphasis is also given to the concept of tracing.

After the necessary preliminaries are disposed of, Section I motivates the study of $\mathfrak{Z}$-filters by discussing the collection of zero-sets and then the Wallman compactification. The concept of realcompactification is also mentioned. Section II introduces the concept of $\mathfrak{Z}$-filters and exhibits many of the elementary facts about these generalized filters.

The beginning of Section III deals with the convergence of $\mathfrak{Z}$-filters. The Frink or Wallman-type compactification of a Tychonoff space is presented in detail. The generalization of this method in constructing realcompactifications in Tychonoff spaces is also discussed. The section closes with the presentation of some tracing results. Finally, Section IV investigates the concept of $\mathfrak{Z}$-continuity.

TABLE OF CONTENTS

I. INTRODUCTION -- 5

II. THE CONCEPT OF $\mathfrak{Z}$-FILTERS ------------------------ 13

III. CONVERGENCE and COMPACTIFICATIONS --------------- 27

IV. $\mathfrak{Z}$ -CONTINUITY ------------------------------------ 56

BIBLIOGRAPHY -- 60

INITIAL DISTRIBUTION LIST ------------------------------ 63

DD FORM 1473 --- 65

4

I. INTRODUCTION

This thesis assumes the reader is familiar with the basics of general topology. Many of the results in this manuscript are contained in or are generalizations from *A Treatise on Realcompactness* by M. Weir [33] and the notes of R. Alo and H. Shapiro [2]. The text *Rings of Continuous Functions* by L. Gillman and M. Jerison [11] provides much of the motivation for this investigation and is referred to frequently. When a precise reference to one of the above works is helpful, the notation ([33], 1.3) is used to specify Section 3 of Chapter 1 of [33].

A *topological space* is an ordered pair (X, τ) where X is a non-empty set and τ is the family of all open subsets of X. Usually the τ is suppressed and (X, τ) is denoted simply by X. When A is a subset of a topological space X, it will be assumed that A is a topological space equipped with the *relative topology* $\tau_A = \{ G \cap A : G \in \tau \}$.

The *power set* of any set X is denoted by $\mathbb{P}(X)$. The *empty set* is represented by ϕ. For a subset A of X and a subcollection $\mathcal{C}$ of $\mathbb{P}(X)$ the *trace of* $\mathcal{C}$ on A is the collection $\{ C \cap A : C \in \mathcal{C}\}$ of subsets of A and is represented by $\mathcal{C} \cap A$. The natural numbers are denoted by $\mathbb{N}$.

The *interior of a subset* A of a topological space X is represented by $\text{int}_X A$, or simply int A when no confusion results. Similarly, the *closure of* A is abbreviated by

$cl_X\, A$ or cl A. The complement of A with respect to X is denoted by $X \setminus A$.

The topological space X is called a T_1- space in case for each $x \varepsilon X$ the singleton set $\{x\}$ is closed. The space X is Hausdorff when for every pair of distinct points $x, y \varepsilon X$ there are disjoint open sets U and V such that $x \varepsilon U$ and $y \varepsilon V$. The space X is completely regular in case for each $x \varepsilon X$ and each closed set F which does not contain x there is a continuous function f from X into the reals such that $f(x) = 0$ and $f(F) \subset \{1\}$. Furthermore X is a Tychonoff space when X is a completely regular T_1- space. Finally X is a normal space in case for each pair of disjoint closed sets F_1 and F_2 there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

In a topological space X a collection $\mathcal{B}$ of closed subsets is called a base for the closed sets if each closed set can be written as the intersection of elements of $\mathcal{B}$. Equivalently $\mathcal{B}$ is a base for the closed sets in case for each closed set $F \subset X$ and each point $x \varepsilon X \setminus F$ there is a basic closed set $B \varepsilon \mathcal{B}$ satisfying $F \subset B$ and $x \notin B$.

The collection of all continuous, real-valued functions on a topological space X is denoted by C(X). The set C(X) can be made into a ring by defining two operations by the formulas: $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ where $f, g \varepsilon C(X)$ and $x \varepsilon X$.

A topological space X is said to be compact in case every open covering of X has a finite subcovering. Equivalently,

X is compact if and only if every family of closed sets with the finite intersection property has non-empty intersection ([15], 5.1).

Gilman and Jerison in ([11], 4) investigate the relationships between a Tychonoff space X that is compact and the associated algebraic ring C(X). They show the known result that <u>two compact Hausdorff spaces are homeomorphic if and only if their respective associated algebraic rings of continuous, real-valued functions are algebraically isomorphic</u>. In other words, the topology of a compact Hausdorff space X is determined by the ring C(X). This result is obtained in the following way.

For a compact space X every maximal ideal in the ring C(X) is of the form $M_x = \{ f \in C(X) : f(x) = 0 \}$ for $x \in X$, and these maximal ideals are distinct for distinct points of X. The collection $\mathfrak{M}$ of all maximal ideals in C(X) is made into a topological space by taking as a base for the closed sets in $\mathfrak{M}$ all sets of the form $\{ M_x : f \in M_x \}$ for $f \in C(X)$. It can be shown that $\mathfrak{M}$ is a well-defined Hausdorff space and that the function g from X into $\mathfrak{M}$ defined by $g(x) = M_x$ for each $x \in X$ is a bijection. Notice that for any $f \in C(X)$, the maximal ideal M_x belongs to the collection $B = \{ M \in \mathfrak{M} : f \in M \}$ if and only if $f(x) = 0$. Hence, $g^{-1}[B] = \{ x \in X : g(x) \in B \} = \{ x \in X : f(x) = 0 \}$. This observation is used to establish that the mapping g is a homeomorphism of X onto $\mathfrak{M}$ ([11], 4.9).

It turns out that sets of the form $\{ x \in X : f(x) = 0 \}$ for $f \in C(X)$, as introduced in the preceding discussion, play

an important role in the study of Tychonoff spaces and deserve further investigation.

Let X be a topological space and let $f \varepsilon C(X)$. The set $Z(f) = \{ x \varepsilon X : f(x) = 0 \}$ is called the zero-set of f. If $A \subset X$, then A is a zero-set in case $A = Z(f)$ for some $f \varepsilon C(X)$. The collection $\{ Z(f) : f \varepsilon C(X) \}$ of all zero-sets on X is denoted by $\mathcal{Z}(X)$. It is clear that each zero-set is closed.

Notice that $Z(f) = Z(|f|) = Z(f^n)$. If $f \equiv 0$ then $Z(f) = X$ and if $f \equiv 1$ then $Z(f) = \emptyset$. The following lemma [11] furnishes some more interesting properties of the zero-sets and provides motivation for some of the generalizations which occur later.

1.1 Lemma. Let X be a topological space and $\mathcal{Z}(X)$ be the collection of zero-sets on X. The following statements are true:

(1) The collection $\mathcal{Z}(X)$ is closed under finite unions.

(2) The collection $\mathcal{Z}(X)$ is closed under countable intersections.

(3) Every zero-set may be represented as a countable intersection of open sets.

(4) The space X is completely regular if and only if $\mathcal{Z}(X)$ is a base for the closed sets.

(5) If X is completely regular, then every neighborhood of a point contains a zero-set neighborhood of the point.

Proof: (1) Let $f, g \varepsilon C(X)$. Since $C(X)$ is a ring, $fg \varepsilon C(X)$. Moreover, it is easy to verify that $Z(f) \cup Z(g) = Z(fg)$. Hence $\mathcal{Z}(X)$ is closed under finite unions.

(2) Let $\{ f_n \varepsilon C(X) : n \varepsilon \mathbb{N} \}$ be a countable collection of elements of $C(X)$. For each $n \varepsilon \mathbb{N}$ let h_n be the constant function $h_n \equiv 2^{-n}$ and define a function g_n by $g_n = \min\{f_n, h_n\}$. From analysis $g_n \varepsilon C(X)$. Furthermore, $g(x) = \sum_{n \varepsilon \mathbb{N}} g_n(x)$ is also in $C(X)$ because $|g_n(x)| \leq 2^{-n}$ so that the above series converges uniformly. It is easy to see that $\{ Z(f_n) : n \varepsilon \mathbb{N} \} = \{ Z(g_n) : n \varepsilon \mathbb{N} \} = Z(g)$. Hence $\mathcal{Z}(X)$ is closed under countable intersections.

(3) Let $f \varepsilon C(X)$ and let $G_n = \{ x \varepsilon X : |f(x)| < 1/n \}$ for each $n \varepsilon \mathbb{N}$. Note that each G_n is an open set. One can verify that $Z(f) = \cap \{ G_n : n \varepsilon \mathbb{N} \}$. Therefore each zero-set may be represented as a countable intersection of open sets; i.e., is a G_δ-set in X.

(4) Let X be completely regular and let F be a closed set and $x \varepsilon X \setminus F$. Then there exists a $f \varepsilon C(X)$ such that $f(x) = 1$ and $f(F) \subset \{0\}$. Hence $F \subset Z(f)$ and $x \notin Z(f)$ so that $\mathcal{Z}(X)$ is a base for the closed sets. Conversely, let $\mathcal{Z}(X)$ be a base for the closed sets and let F be a closed set with $x \varepsilon X \setminus F$. Then there is a $g \varepsilon C(X)$ such that $F \subset Z(g)$ and $x \notin Z(g)$. So $g(x) = c$ where c is a non-zero constant. Define the function $f = \frac{1}{c} g$. Then $f \varepsilon C(X)$ and it is easily verified that $f(x) = 1$ and $f(F) \subset \{0\}$. Hence X is completely regular.

(5) Let $x \in X$ and let $N(x)$ be a neighborhood of x. Then there is a closed set F such that $X \setminus N(x) \subset F$ and $x \notin F$. Since X is completely regular, there is a $f \in C(X)$ such that $f(x) = 1$ and $f(F) \subset \{0\}$. Let $h, k \in C(X)$ be the functions $h \equiv 2/3$ and $k \equiv 1/3$. Since

$$A = \{ x \in X : f(x) \geq 2/3 \} = Z(\min \{f-h, 0\}) \text{ and}$$

$$B = \{ x \in X : f(x) \leq 1/3 \} = Z(\max \{f-k, 0\}),$$ the sets A and B are disjoint zero-set neighborhoods of $\{ x \}$ and F respectively. Furthermore, $A \subset N(x)$ because $X \setminus N(x) \subset B$ and $B \cap A = \emptyset$. Thus, $x \in \operatorname{int} A \subset A \subset N(x)$.

Prior to the introduction of zero-sets, an interesting property of compact spaces was discussed. The compact spaces enjoy other important properties. For instance, continuous real-valued functions assume their infimums and supremums. It is therefore often desirable to embed a space into a compact space. A <u>compactification of a space</u> X is an ordered pair (f,Y) where Y is a compact space and f is a homeomorphism from X onto a dense subspace of Y. A compactification is called <u>Hausdorff</u> in case Y is a Hausdorff space.

Wallman [32] developed a technique for constructing a Hausdorff compactification of any normal space. An outline of the essential steps in his procedure ([15], 5.R) provides insight into later developments of this paper.

Let X be a T_1- space, let $\mathfrak{F}$ be the family of all closed subsets of X and let $\omega(\mathfrak{F})$ be the collection of all subfamilies

$\mathfrak{F}$ of $\mathfrak{Z}$ which have the finite intersection property and are maximal in $\mathfrak{Z}$ relative to this property. If $\mathfrak{F} \varepsilon \omega(\mathfrak{Z})$, then $\mathfrak{F}$ is closed under finite intersections. For each closed subset F of X let $F^* = \{\mathfrak{F} \varepsilon\ \omega(\mathfrak{Z}) : F \varepsilon \mathfrak{F}\}$. Then $\omega(\mathfrak{Z})$ can be made into a compact topological space by taking as a base for the closed sets the collection $\{F^* : F \varepsilon \mathfrak{Z}\}$. For each $x \varepsilon X$, let $f(x) = \{F \varepsilon \mathfrak{Z} : x \varepsilon F\}$. It can be shown that f is a homeomorphism of X onto a dense subspace of $\omega(\mathfrak{Z})$. Moreover, if X is normal then $\omega(\mathfrak{Z})$ is Hausdorff.

Frink [10] generalized Wallman's method by constructing Hausdorff compactifications for arbitrary Tychonoff spaces. He used the notion of a <u>normal base</u> $\mathfrak{Z}$ (which will be defined later) to construct the space $\omega(\mathfrak{Z})$ of all subfamilies $\mathfrak{F}$ of $\mathfrak{Z}$ which have the finite intersection property and are maximal in $\mathfrak{Z}$ relative to this property. Frink's construction will be presented in detail in Section III and motivates many of the concepts to be presented in this manuscript.

The concept of a <u>realcompact space</u> (originally known as a Q-space) is due to E. Hewitt [14]. The theory of realcompact spaces [33] is in many ways analogous to the theory of compact spaces. In fact, the realcompact spaces are determined by the associated algebraic ring C(X) just as the compact spaces are determined by the subring C*(X) of all bounded functions in C(X). (Note that for a compact space X the rings C(X) and C*(X) coincide since every continuous, real-valued function on a compact space is bounded.) There are many equivalent

formulations for the concept of a realcompact space (See [33], 2.5.1, 2.5.4, 2.5.5, 2.5.9, 2.6.4, 2.6.11, 3.11.6, 4.13.25). This manuscript will define a realcompact space as a Tychonoff space X that can be homeomorphically embedded as a closed subspace in a product of real lines. An equivalent characterization of realcompactness will be given later on in this work.

In [3] Alo and Shapiro use a variation of Frink's notion of a normal base in order to construct realcompactifications of Tychonoff spaces that are of the Wallman or Frink type. Their considerations will also motivate many of the definitions and results to be presented in this manuscript.

II. THE CONCEPT OF $\mathfrak{Z}$-FILTERS

The theory of filters plays an important role in topology. A generalization of the filter concept provides a useful tool for the study of compactness and realcompactness as was mentioned earlier. This thesis assembles and extends some of the work that has been done with generalized filters and relates this work to compactifications and realcompactifications. The following definition is fundamental.

2.1 Definition. If X is a non-empty set and if $\mathfrak{Z} \subset \mathbb{P}(X)$ is closed under finite intersections, then a collection $\mathfrak{F}$ of elements in $\mathfrak{Z}$ is a $\mathfrak{Z}$-filter on X when the following conditions are satisfied:

(1) The collection $\mathfrak{F}$ is non-empty and $\phi \not\in \mathfrak{F}$.

(2) If A and B belong to $\mathfrak{F}$, then $A \cap B \in \mathfrak{F}$.

(3) If $A \in \mathfrak{Z}$ and A contains an element of $\mathfrak{F}$, then $A \in \mathfrak{F}$.

If $\mathfrak{Z} = \mathbb{P}(X)$, then $\mathfrak{Z}$-filters are simply the familiar Bourkaki filters. Since the zero-sets are closed under countable intersections, the zero-sets may be used for $\mathfrak{Z}$ and the corresponding zero-set-filters (or $\mathcal{Z}$-filters) are then considered. Throughout the remainder of this paper, it will be assumed that the collection $\mathfrak{Z}$ is closed under finite intersections.

2.2 *Definition*. A collection $\mathcal{B}$ of members of a $\mathfrak{Z}$ -filter $\mathfrak{F}$ is called a *base for* $\mathfrak{F}$ if for each $A \in \mathfrak{F}$ there is an element of $\mathcal{B}$ contained in A.

2.3 *Lemma*. *Let* $\mathcal{B}$ *be a subcollection of* $\mathfrak{Z}$. *The collection* $\mathcal{B}$ *is a base for a* $\mathfrak{Z}$*-filter* $\mathfrak{F}$ *if and only if both of the following hold:*

(1) *The collection* $\mathcal{B}$ *is non-empty and* $\phi \notin \mathcal{B}$.

(2) *The intersection of any two elements of* $\mathcal{B}$ *contains a member of* $\mathcal{B}$.

Proof: Let $\mathcal{B}$ be a base for a $\mathfrak{Z}$-filter $\mathfrak{F}$. Then $\mathcal{B}$ is non-empty by definition of a $\mathfrak{Z}$-filter base and the fact that $\mathfrak{F}$ is non-empty. Moreover, $\mathcal{B} \subset \mathfrak{F}$ implies $\phi \in \mathcal{B}$. Also, from $\mathcal{B} \subset \mathfrak{F}$ and the definition of $\mathfrak{Z}$ -filter, $B_1, B_2 \in \mathcal{B}$ implies $B_1 \cap B_2 \in \mathfrak{F}$. Then by definition there is a member of $\mathcal{B}$ contained in $B_1 \cap B_2$.

Conversely, suppose (1) and (2) hold. Consider $\mathfrak{F} = \{ A \in \mathfrak{Z} : B \subset A, B \in \mathcal{B} \}$. If $\mathfrak{F}$ is a $\mathfrak{Z}$-filter, then clearly $\mathcal{B}$ is a base for $\mathfrak{F}$. From (1) and the fact that $\mathcal{B} \subset \mathfrak{F}$, the collection $\mathfrak{F}$ is non-empty and $\phi \notin \mathfrak{F}$. The intersection of any two elements of $\mathfrak{F}$ contains the intersection of two elements of $\mathcal{B}$ which in turn contains a member of $\mathcal{B}$ by (2). It follows that the intersection of any two elements of $\mathfrak{F}$ must belong to $\mathfrak{F}$. Finally, let $Z \in \mathfrak{Z}$ such that $Z \supset A$ where $A \in \mathfrak{F}$. Th.n there is a $B \in \mathcal{B}$ such that $Z \supset A \supset B$. Hence $Z \in \mathfrak{F}$.

Therefore, $\mathfrak{F}$ is a $\mathcal{Z}$-filter with base $\mathcal{B}$.

2.3.1 *Remark.*

If $\mathcal{B}$ is a base for a $\mathcal{Z}$-filter, then the $\mathcal{Z}$-filter $\mathfrak{F} = \{A \varepsilon \mathcal{Z} : B \subset A \text{ for some } B \varepsilon \mathcal{B}\}$ of the preceding proof is called *the $\mathcal{Z}$-filter generated by $\mathcal{B}$*. Moreover, if $\mathcal{B}$ is a base for the $\mathcal{Z}$-filter $\mathfrak{F}$, then $\mathfrak{F}$ is the $\mathcal{Z}$-filter generated by $\mathcal{B}$.

It is easy to show from the definitions that if $\mathfrak{G}$ is a subcollection of $\mathcal{Z}$, then there is a $\mathcal{Z}$-filter containing $\mathfrak{G}$ if and only if every finite subcollection of $\mathfrak{G}$ has non-empty intersection.

The following lemma will be used frequently.

2.4 *Lemma. Let $\mathfrak{F}$ be a $\mathcal{Z}$-filter on X and let $A \varepsilon \mathcal{Z}$. Then $\mathfrak{F} \cup \{A\}$ is contained in some $\mathcal{Z}$-filter if and only if A meets every member of $\mathfrak{F}$.*

Proof: The necessity is clear because the empty set cannot belong to any $\mathcal{Z}$-filter. On the other hand suppose A meets every member of $\mathfrak{F}$ and let $\mathcal{B} = \mathfrak{F} \cap A$. By hypothesis $\emptyset \notin \mathcal{B}$, and since $\mathfrak{F}$ is non-empty, the same holds true of $\mathcal{B}$. The intersection of two elements of $\mathcal{B}$ belongs to $\mathcal{B}$ because $\mathfrak{F}$ is closed under finite intersections. Hence by (2.3), $\mathcal{B}$ generates a $\mathcal{Z}$-filter $\mathfrak{G}$ satisfying $\mathfrak{F} \subset \mathfrak{G}$ and $A \varepsilon \mathfrak{G}$.

An examination of the trace of $\mathcal{Z}$-filters reveals difficulties not present in the study of Bourbaki filters. One

question which arises is what should be taken as the distinguished collection of subsets of A. Suppose for the moment that the collection $\mathfrak{Z} \cap A$ is selected. After all, the power set of A is given by $\mathbb{P}(X) \cap A$. Recall that for a Bourbaki filter $\mathfrak{F}$ and a subset A of X the trace of $\mathfrak{F}$ on A is a filter if and only if A meets every member of $\mathfrak{F}$. The following example illustrates that this result need not hold for $\mathfrak{Z}$ -filters.

Let X be the natural numbers, let the collection $\mathfrak{Z}$ be given by $\{ \{1\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \emptyset\}$ and let $A = \{1,4\}$. Notice that $\mathfrak{Z}$ is closed under finite intersections and that $\mathfrak{Z} \cap A = \{ \{1\}, \{1,4\}, \emptyset\}$. The collection $\mathfrak{F} = \{ \{1,2,3\} \}$ is a $\mathfrak{Z}$-filter but $\mathfrak{F} \cap A = \{\{1\}\}$ is clearly not a $\mathfrak{Z} \cap A$ -filter.

Next, suppose the collection of zero-sets is the distinguished collection of interest. Let X be the real line and let A be defined as $X \setminus \{a\}$ where a is a point in X. Notice that the set $\{ x \in X : x > a \}$ is a zero-set of A determined by the continuous function f on A which takes $x > a$ into 0 and $x < a$ into 1. However, f does not have a continuous extension to all of X. In other words, there exists a zero-set on A determined by a continuous function f which is not obtained by the intersection of A with the zero-set of a function on the whole space.

This example suggests the blanket assumption that $\mathfrak{Z} \cap A$ be taken as the distinguished collection of subsets of A

should not be made. Throughout the remainder of this paper, $\mathfrak{Z}_A$ will denote simply an arbitrary subcollection of $\mathbb{P}(A)$. The following lemmas investigate the problem of tracing with $\mathfrak{Z}$-filters. Additional results will be obtained in Section III. The following definition will be useful.

2.5 Definition. Let X be a set and let $\mathfrak{Z} \subset \mathbb{P}(X)$. The collection $\mathfrak{Z}$ is a ring of sets in case it is closed under finite unions and also finite intersections.

The power set of any set is clearly a ring of sets. From (1.1) the collection of zero-sets for any X is also a ring of sets.

2.6 Lemma. Let $A \subset X$ and let $\mathfrak{F}$ be a $\mathfrak{Z}$-filter with base $\mathcal{B}$ such that $\mathfrak{Z}_A \supset \mathfrak{F} \cap A$. Then the following statements are equivalent:

(1) The set A meets every member of $\mathfrak{F}$.

(2) The collection $\mathfrak{F} \cap A$ generates a $\mathfrak{Z}_A$-filter.

(3) The collection $\mathcal{B} \cap A$ is a $\mathfrak{Z}_A$-filter base.

Proof: (1) implies (2). The collection $\mathfrak{F} \cap A$ is non-empty by hypothesis since $\mathfrak{F}$ is non-empty. By (1) the empty set does not belong to $\mathfrak{F} \cap A$. Finally the intersection of two members of $\mathfrak{F} \cap A$ belongs to $\mathfrak{F} \cap A$ since $\mathfrak{F}$ is closed under finite intersections. Hence, by (2.3) the collection $\mathfrak{F} \cap A$ generates a $\mathfrak{Z}_A$-filter.

(2) implies (3). Since $\mathcal{B}$ is non-empty, the same is true of $\mathcal{B} \cap A$. The empty set is not in $\mathcal{B} \cap A$ because $\mathcal{B} \subset \mathfrak{T}$ and $\mathfrak{T} \cap A$ generates a $\mathfrak{Z}_A$-filter. The intersection of two elements of $\mathcal{B} \cap A$ contains a member of $\mathcal{B} \cap A$ since the intersection of any two elements of $\mathcal{B}$ contains an element of $\mathcal{B}$.

(3) implies (1). Any $F \varepsilon \mathfrak{T}$ contains a $B \varepsilon \mathcal{B}$ by definition of filter base. Hence $F \cap A$ is non-empty because $F \cap A \supset B \cap A$, and $B \cap A$ is non-empty by (3).

2.7 *Lemma*. *Let* $A \subset X$, $\mathfrak{Z}$ *be a ring of sets*, $\mathfrak{Z}_A \subset \mathfrak{Z} \cap A$, *and let* $\mathfrak{T}$ *be a* $\mathfrak{Z}$-*filter with base* $\mathcal{B}$ *satisfying* $\mathfrak{T} \cap A \subset \mathfrak{Z}_A$. *If* $\mathcal{B} \cap A$ *is a* $\mathfrak{Z}_A$-*filter base on* A, *then* $\mathcal{B} \cap A$ *generates the* $\mathfrak{Z}_A$-*filter* $\mathfrak{T} \cap A$.

Proof: Let $\mathfrak{G}$ be the $\mathfrak{Z}_A$-filter generated by $\mathcal{B} \cap A$. For any $F \varepsilon \mathfrak{T}$ there is a $B \varepsilon \mathcal{B}$ such that $F \supset B$. Then $F \cap A \supset B \cap A$ so that $F \cap A \varepsilon \mathfrak{G}$. Hence $\mathfrak{T} \cap A \subset \mathfrak{G}$. On the other hand, let $G \varepsilon \mathfrak{G}$. Then there is a $B \varepsilon \mathcal{B}$ such that $G \supset B \cap A$. Furthermore, since $\mathfrak{Z}_A \subset \mathfrak{Z} \cap A$, there is a $Z \varepsilon \mathfrak{Z}$ such that $G = Z \cap A$. Hence $Z \cap A \supset B \cap A$. Since $\mathfrak{Z}$ is a ring of sets, $Z \cup B \varepsilon \mathfrak{Z}$. Moreover, $Z \cup B$ is necessarily in $\mathfrak{T}$. Hence $(Z \cup B) \cap A = (Z \cap A) \cup (B \cap A) = Z \cap A = G$ is an element of $\mathfrak{T} \cap A$. That is, $G \subset \mathfrak{T} \cap A$. Therefore, $\mathfrak{G} = \mathfrak{T} \cap A$.

The proof of the following corollary is immediate from (2.6) and (2.7).

2.8 *Corollary*. *Let* $A \subset X$, $\mathcal{Z}$ *be a ring of sets*, $\mathcal{Z}_A \subset \mathcal{Z} \cap A$, *and let* $\mathcal{T}$ *be a* $\mathcal{Z}$*-filter with base* $\mathcal{B}$ *satisfying* $\mathcal{T} \cap A \subset \mathcal{Z}_A$. *The following statements are equivalent:*

(1) *The set* A *meets every member of* $\mathcal{T}$.

(2) *The collection* $\mathcal{B} \cap A$ *is a* $\mathcal{Z}_A$*-filter base.*

(3) *The collection* $\mathcal{T} \cap A$ *is a* $\mathcal{Z}_A$*-filter.*

2.9 *Lemma*. *Let* $A \subset X$ *and let* $\mathcal{G}$ *be a* $\mathcal{Z}_A$*-filter such that for every* $G \varepsilon \mathcal{G}$ *there is a* $Z \varepsilon \mathcal{Z}$ *such that* $Z \cap A = G$. *If for every* $Z \varepsilon \mathcal{Z}$ *that contains an element of* $\mathcal{G}$ *it is true that* $Z \cap A$ *belongs to* $\mathcal{G}$, *then there is a* $\mathcal{Z}$*-filter* $\mathcal{T}$ *such that* $\mathcal{T} \cap A = \mathcal{G}$.

Proof: The collection $\mathcal{T} = \{ Z \varepsilon \mathcal{Z} : Z \supset G,\ G \varepsilon \mathcal{G}\}$ is non-empty because $\mathcal{G}$ is non-empty and for each $G \varepsilon \mathcal{G}$ there is a $Z \varepsilon \mathcal{Z}$ with $Z \supset G$. The empty set is not in $\mathcal{T}$ since $\phi \notin \mathcal{G}$. If $Z_1, Z_2 \varepsilon \mathcal{T}$, then there are $G_1, G_2 \varepsilon \mathcal{G}$ such that $Z_1 \supset G_1$ and $Z_2 \supset G_2$. Hence $Z_1 \cap Z_2 \supset G_1 \cap G_2$ so that $Z_1 \cap Z_2 \varepsilon \mathcal{T}$. Furthermore, if an element of $\mathcal{Z}$ contains an element of $\mathcal{T}$, it must contain a member of $\mathcal{G}$ and thus also belong to $\mathcal{T}$. Hence, $\mathcal{T}$ is a $\mathcal{Z}$-filter. Now, if $Z \varepsilon \mathcal{T}$ then $Z \supset G$ for some $G \varepsilon \mathcal{G}$. Then by hypothesis, $Z \cap A \ \varepsilon \mathcal{G}$. Hence $\mathcal{T} \cap A \subset \mathcal{G}$. On the other hand, if $G \varepsilon \mathcal{G}$ then there is a $Z \varepsilon \mathcal{Z}$ such that $Z \cap A = G$. Hence $Z \varepsilon \mathcal{T}$ by definition. Therefore $\mathcal{T} \cap A = \mathcal{G}$.

2.10 *Corollary*. *If* $A \subset X$ *and if* $\mathcal{G}$ *is a* $\mathcal{Z} \cap A$*-filter, then there exists a* $\mathcal{Z}$*-filter* $\mathcal{T}$ *such that* $\mathcal{T} \cap A = \mathcal{G}$.

Proof: For each $G \in \mathcal{G}$ there is a $Z \in \mathcal{Z}$ such that $Z \cap A = G$, and for each $Z \in \mathcal{Z}$ it is true that $Z \cap A \in \mathcal{Z} \cap A$. If $Z \in \mathcal{Z}$ contains a member G of $\mathcal{G}$, then $Z \cap A \supset G$ implies that $Z \cap A$ is an element of $\mathcal{G}$. Now apply (2.9).

A $\mathcal{Z}$-filter $\mathcal{U}$ is called a $\mathcal{Z}$-*ultrafilter* provided it is a maximal $\mathcal{Z}$-filter with respect to the partial ordering of set inclusion in the collection of all $\mathcal{Z}$-filter on X. In other words, if $\mathcal{F}$ is a $\mathcal{Z}$-filter containing $\mathcal{U}$ then $\mathcal{F} = \mathcal{U}$. The following result carries over from Bourbaki filters.

2.11 *Lemma*. *Every $\mathcal{Z}$-filter is contained in a $\mathcal{Z}$-ultrafilter.*

Proof: Let $\mathcal{F}$ be a $\mathcal{Z}$-filter and denote the collection of $\mathcal{Z}$-filters containing $\mathcal{F}$ by $\mathfrak{F}(X)$. Then $\mathfrak{F}(X)$ is partially ordered by set inclusion. If Φ is any chain in $\mathfrak{F}(X)$, then the union $\cup\Phi$ of the $\mathcal{Z}$-filters belonging to Φ also belongs to $\mathfrak{F}(X)$. This assertion is established as follows. First, $\cup\Phi$ is non-empty because Φ is non-empty. Also, $\emptyset \notin \cup\Phi$ because the empty set does not belong to any $\mathcal{Z}$-filter in Φ. Next, for any two elements Z_1 and Z_2 of $\cup\Phi$ there are filters $\mathcal{F}$ and $\mathcal{G}$ in Φ such that $Z_1 \varepsilon \mathcal{F}$ and $Z_2 \varepsilon \mathcal{G}$. Since Φ is a chain, it may be assumed without loss of generality that $\mathcal{F} \subset \mathcal{G}$. Hence $Z_1 \cap Z_2$ belongs to $\mathcal{G}$ and therefore also belongs to $\cup\Phi$. Finally, for any $Z \varepsilon \mathcal{Z}$ which contains an element $Z_1 \varepsilon \cup\Phi$, it is the case that Z belongs to the $\mathcal{Z}$-filter $\mathcal{F}$ satisfying $Z_1 \varepsilon \mathcal{F}$ and $\mathcal{F} \varepsilon \Phi$. Hence the set Z is in $\cup\Phi$. Thus every chain Φ in $\mathfrak{F}(X)$ has an

upper bound in $\mathfrak{F}(X)$. By Zorn's Lemma the collection $\mathfrak{F}(X)$ has a maximal element $\mathcal{U}$. Suppose a $\mathfrak{Z}$-filter $\mathfrak{G}$ contains $\mathcal{U}$. Then $\mathfrak{G}$ must contain $\mathfrak{F}$ and hence belong to $\mathfrak{F}(X)$. Since $\mathcal{U}$ is maximal in $\mathfrak{F}(X)$, the $\mathfrak{Z}$-filter $\mathfrak{G}$ must be precisely $\mathcal{U}$. Therefore, $\mathcal{U}$ is a $\mathfrak{Z}$-ultrafilter which contains $\mathfrak{F}$ which completes the proof.

2.12 Lemma. Let $\mathfrak{Z}$ be a ring of sets such that $A \varepsilon \mathfrak{Z}$, let $\mathfrak{Z}_A \subset \mathfrak{Z} \cap A$, and let $\mathfrak{F}$ be a $\mathfrak{Z}$-ultrafilter satisfying $\mathfrak{F} \cap A \subset \mathfrak{Z}_A$. Then $\mathfrak{F} \cap A$ is a $\mathfrak{Z}_A$-filter if and only if $A \varepsilon \mathfrak{F}$. In this case $\mathfrak{F} \cap A$ is, in fact, a $\mathfrak{Z}_A$-ultrafilter.

Proof: The set A meets every member of $\mathfrak{F}$ because $\mathfrak{F} \cap A$ is a $\mathfrak{Z}_A$-filter. By (2.4) the collection $\mathfrak{F} \cup \{A\}$ is contained in a $\mathfrak{Z}$-filter $\mathfrak{G}$. But $\mathfrak{F}$ is maximal so $\mathfrak{F} = \mathfrak{G}$. Finally, $A \varepsilon \mathfrak{F}$ because $A \varepsilon \mathfrak{G}$. Conversely, if $A \varepsilon \mathfrak{F}$ then A meets every member of $\mathfrak{F}$ so that $\mathfrak{F} \cap A$ is a $\mathfrak{Z}_A$-filter by (2.8). Now suppose there is a $\mathfrak{Z}_A$-filter $\mathfrak{G}$ containing $\mathfrak{F} \cap A$ and let $G \varepsilon \mathfrak{G}$. By hypothesis there is a set $Z \varepsilon \mathfrak{Z}$ such that $Z \cap A = G$. For every $F \varepsilon \mathfrak{F}$ the intersection $(F \cap A) \cap (Z \cap A)$ is non-empty because $F \cap A$ belongs to $\mathfrak{G}$. It follows that $F \cap Z \neq \phi$. Once again (2.4) may be applied so that $\mathfrak{F} \cup \{Z\}$ is contained in a $\mathfrak{Z}$-filter $\mathcal{V}$. As before it follows that $\mathfrak{F} = \mathcal{V}$ so $Z \varepsilon \mathfrak{F}$. Therefore, $Z \cap A = G$ belongs to $\mathfrak{F} \cap A$ so $\mathfrak{G} = \mathfrak{F} \cap A$. It then follows that $\mathfrak{F} \cap A$ is a $\mathfrak{Z}_A$-ultrafilter.

Recall that a space is compact if and only if every family of closed sets with the finite intersection property has a

non-empty intersection. If $\mathfrak{Z}$ is a collection of closed sets, for example the zero-sets, it follows that X will be compact only if each $\mathfrak{Z}$-filter $\mathcal{F}$ is such that $\cap\{ Z\varepsilon\mathfrak{Z} : Z\varepsilon \mathcal{F} \} \neq \emptyset$ because $\mathcal{F}$ has the finite intersection property. This observation leads to the next definition together with the lemma and corollary which follow.

2.13 Definition. A $\mathfrak{Z}$-filter $\mathcal{F}$ is said to be fixed provided $\cap\{ Z\varepsilon\mathfrak{Z} : Z\varepsilon\mathcal{F}\} \neq \emptyset$; otherwise, $\mathcal{F}$ is said to be free.

2.14 Lemma. A Tychonoff space X is compact if and only if every zero-set-ultrafilter ($\mathcal{Z}$-ultrafilter) on X is fixed.

Proof: Since any $\mathcal{Z}$-ultrafilter $\mathcal{U}$ is a collection of closed sets with the finite intersection property, $\mathcal{U}$ must be fixed whenever X is compact. Now, suppose that every $\mathcal{Z}$-ultrafilter is fixed. Let $\mathcal{C}$ be a family of closed sets with the finite intersection property. By (1.1) the zero-sets are a base for the closed sets so that each member of $\mathcal{C}$ may be represented as the intersection of a collection of zero-sets. Let $\mathcal{B} = \{ Z\varepsilon\ \mathcal{Z}(X) : Z\varepsilon\{Z_\lambda\varepsilon\ \mathcal{Z}(X) : \lambda\varepsilon\Lambda \}\}$ where $C = \cap\{ Z_\lambda : \lambda\varepsilon\Lambda \}$ for some $C\varepsilon\mathcal{C}$. Now let $\mathcal{B}'$ consist of all finite intersections of members of $\mathcal{B}$. The collection $\mathcal{B}'$ is a $\mathcal{Z}$-filter base. This assertion is true because the finite intersection property of $\mathcal{C}$ together with the fact that each member of $\mathcal{B}$ contains an element of $\mathcal{C}$ imply that $\mathcal{B}'$ is a non-empty collection such that $\emptyset \notin \mathcal{B}'$. Moreover, by the construction of $\mathcal{B}'$ the intersection of two members of $\mathcal{B}'$ is in $\mathcal{B}'$.

Then (2.3) may be applied. It follows that $\mathcal{B}'$ is contained in some $\mathcal{Z}$-ultrafilter $\mathcal{U}$ which is fixed by hypothesis. Finally, $\cap\{ C : C \varepsilon \mathcal{C}\} \neq \phi$ because $\cap\{ Z\varepsilon \mathcal{Z}(X) : Z\varepsilon \mathcal{U}\} \subset \{ Z\varepsilon \mathcal{Z}(X) : Z\varepsilon \mathcal{B}' \} \subset \{ Z\varepsilon \mathcal{Z}(X) : Z\varepsilon \mathcal{B}\} = \cap\{ C : C\varepsilon \mathcal{C}\}$ which completes the proof.

2.15 <u>Corollary</u>. <u>A Tychonoff space</u> X <u>is compact if and only if every</u> $\mathcal{Z}$-<u>filter is fixed</u>.

<u>Proof</u>: The proof simply consists of using (2.14) together with the fact that every $\mathcal{Z}$-filter is contained in a $\mathcal{Z}$-ultrafilter.

In Section I the Wallman compactification was discussed. Observe that the collection $\omega(\mathfrak{Z})$ mentioned there is actually the collection of all $\mathfrak{Z}$-ultrafilters on X where $\mathfrak{Z}$ consists of all the closed subsets of X. Moreover, the function f from X into $\omega(\mathfrak{Z})$ defined by $f(x) = \{ Z\varepsilon \mathfrak{Z} : x\varepsilon Z \}$ clearly maps each point of X into a fixed $\mathfrak{Z}$-ultrafilter.

As has been noted, Frink [10] generalized Wallman's method to obtain compactifications of arbitrary Tychonoff spaces. His approach was to impose certain properties on $\mathfrak{Z} \subset \mathcal{P}(X)$ that are analogous to those enjoyed by the collection of closed sets in a normal space. It is then possible to make $\omega(\mathfrak{Z})$ into a compact Hausdorff space such that the points of X can be naturally identified with the fixed $\mathfrak{Z}$-ultrafilters. Furthermore, it can be shown that $\omega(\mathfrak{Z})$ contains a dense homeomorphic image of X.

When X is a T_1-space, each point in X is a member of the collection of all closed sets. Moreover, if $x \varepsilon X$ and $Z \subset X$ is a closed set such that $x \notin Z$, then there is a closed set (namely $\{x\}$) such that $\{x\} \cap Z = \emptyset$. These facts are used in the Wallman compactification to show that for any $x \varepsilon X$ the collection $\{ Z : Z \text{ is closed}, x \varepsilon Z \}$ is in $\omega(\mathfrak{Z})$. This observation motivates the introduction of the following concepts which enable a similar result to be obtained for a more arbitrary $\mathfrak{Z}$.

2.16 <u>Definition</u>. For a non-empty set X, a non-empty collection $\mathfrak{Z} \subset \mathbb{P}(X)$ is said to be $\mathfrak{Z}$-<u>disjunctive</u> if for each set $Z \varepsilon \mathfrak{Z}$ and point $x \notin Z$ there exists a $Z_1 \varepsilon \mathfrak{Z}$ such that $x \varepsilon Z_1$ and $Z \cap Z_1 = \emptyset$. If X is a topological space, then $\mathfrak{Z}$ is said to be <u>disjunctive</u> in case for each closed set $F \subset X$ and point $x \notin F$ there exists a $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon Z$ and $Z \cap F = \emptyset$.

Note that if $\mathfrak{Z}$ is a disjunctive collection of closed sets of X, then $\mathfrak{Z}$ is $\mathfrak{Z}$-disjunctive. It is clear that in a T_1-space the collection of all closed sets is disjunctive as well as $\mathfrak{Z}$-disjunctive. The collection $\mathcal{Z}(X)$ of all zero-sets on a Tychonoff space X is disjunctive and since such sets are closed, $\mathcal{Z}(X)$ is also $\mathfrak{Z}$-disjunctive. Also, note that the power set of X is $\mathfrak{Z}$-disjunctive, and if X is a T_1-space then the power set is disjunctive. The importance of these results is revealed in the following propositions.

2.17 Lemma. *If the collection* $\mathfrak{Z}$ *is* $\mathfrak{Z}$*-disjunctive then for any point* $x \varepsilon X$ *the collection* $\mathfrak{T}_x = \{ Z \varepsilon \mathfrak{Z} : x \varepsilon Z \}$ *is a fixed* $\mathfrak{Z}$*-ultrafilter. Throughout the remainder of this paper the notation* $\mathfrak{T}_x$ *will refer to this particular* $\mathfrak{Z}$ *-ultrafilter.*

Proof: Let $Z \varepsilon \mathfrak{Z}$ and suppose $Z \notin \mathfrak{T}_x$. Then $x \notin Z$ so there exists a $Z_1 \varepsilon \mathfrak{Z}$ such that $x \varepsilon Z_1$ and $Z \cap Z_1 = \phi$ by the $\mathfrak{Z}$ -disjunctive property. It follows that $\mathfrak{T}_x$ is non-empty. The empty set is not in $\mathfrak{T}_x$ because the point x belongs to every element of $\mathfrak{T}_x$. The intersection of any two members of $\mathfrak{T}_x$ contains the point x and therefore belongs to $\mathfrak{T}_x$. For any $Z \varepsilon \mathfrak{Z}$ that contains an element of $\mathfrak{T}_x$ the point x must belong to Z and hence $Z \varepsilon \mathfrak{T}_x$. Clearly, $x \varepsilon \cap \{Z \varepsilon \mathfrak{Z} : Z \varepsilon \mathfrak{T}_x\}$.

Now suppose that $\mathfrak{B}$ is a $\mathfrak{Z}$-filter which contains $\mathfrak{T}_x$. If B is a member of $\mathfrak{B}$ and $B \notin \mathfrak{T}_x$, then $x \notin B$. Hence there is a $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon Z$ and $Z \cap B = \phi$. Thus, Z belongs to $\mathfrak{T}_x$ and hence to $\mathfrak{B}$. This contradicts the finite intersection property of $\mathfrak{B}$. Hence $\mathfrak{B} = \mathfrak{T}_x$ which completes the proof.

2.18 Lemma. *Let the collection* $\mathfrak{Z}$ *be* $\mathfrak{Z}$*-disjunctive and let* $\mathfrak{U}$ *be a* $\mathfrak{Z}$*-ultrafilter. Then* $\mathfrak{U}$ *is fixed if and only if* $\mathfrak{U} = \mathfrak{T}_x$ *for some* $x \varepsilon X$.

Proof: Let $\mathfrak{U}$ be fixed and let $x \varepsilon \cap \{ Z \varepsilon \mathfrak{Z} : Z \varepsilon \mathfrak{U}\}$. Since $\mathfrak{Z}$ is $\mathfrak{Z}$ -disjunctive, $\mathfrak{T}_x$ is a $\mathfrak{Z}$-ultrafilter. For any $Z \varepsilon \mathfrak{U}$ the point $x \varepsilon Z$ so that $Z \varepsilon \mathfrak{T}_x$. It follows that $\mathfrak{U} \subset \mathfrak{T}_x$. Hence $\mathfrak{U} = \mathfrak{T}_x$ since $\mathfrak{U}$ is a $\mathfrak{Z}$-ultrafilter. The proof of the converse is immediate.

Whenever $\mathfrak{Z}$ is $\mathfrak{Z}$-disjunctive or whenever $\mathfrak{Z}$ is a disjunctive collection of closed sets, the association of a point $x \varepsilon X$ with the $\mathfrak{Z}$-ultrafilter $\mathfrak{T}_x$ in $\omega(\mathfrak{Z})$ seems quite natural. If this association is to be used to define a mapping which embeds X into $\omega(\mathfrak{Z})$, then this association must be one-to-one. These considerations motivate some of the definitions which follow in the next section.

III. CONVERGENCE and COMPACTIFICATIONS

The concept of the convergence of Bourbaki filters, or equivalently that of the convergence of nets, plays a major role in the study of continuity, compactness, and realcompactness. The idea of $\mathfrak{Z}$-filter convergence is utilized in obtaining Wallman-type compactifications and realcompactifications. This section examines the convergence properties of $\mathfrak{Z}$-filters and exhibits the results that are useful in obtaining these compactifications and realcompactifications. Recall that $\mathfrak{Z}$ is always closed under finite intersections.

3.1 Definition. Let $\mathcal{B}$ be a $\mathfrak{Z}$-filter base on a topological space X and let $x \varepsilon X$. The point x is said to be a limit point of $\mathcal{B}$ in case for every neighborhood N(x) of x there exists a $B \varepsilon \mathcal{B}$ such that $B \subset N(x)$. In this case $\mathcal{B}$ is said to converge to x. The point x is said to be a cluster point of $\mathcal{B}$ if for each neighborhood N(x) of x, $N(x) \cap B \neq \phi$ for all $B \varepsilon \mathcal{B}$.

The easy proofs of the following facts are omitted. If $\mathfrak{F}$ is a $\mathfrak{Z}$-filter with base $\mathcal{B}$, then the point $x \varepsilon X$ is a limit point (respectively, cluster point) of $\mathfrak{F}$ if and only if x is a limit point (respectively, cluster point) of $\mathcal{B}$. Let $\mathfrak{F}$ and $\mathfrak{G}$ be $\mathfrak{Z}$-filters with $\mathfrak{F} \subset \mathfrak{G}$. If the point $x \varepsilon X$ is a limit point of $\mathfrak{F}$, then x is a limit point of $\mathfrak{G}$. Also, if x is a cluster point of $\mathfrak{G}$, then x is a cluster point of $\mathfrak{F}$. Finally, every fixed $\mathfrak{Z}$-filter has a cluster point.

Recall that the collection of all neighborhoods of a given point x is a Bourbaki filter. Therefore, in the theory of Bourbaki filters, there always exists a Bourbaki filter converging to the point x for any $x \varepsilon X$. However, this is not necessarily the case for $\mathfrak{Z}$-filters.

Consider the real numbers under the discrete topology. Let x and y be real numbers and let $\mathfrak{Z}$ consist of those subsets of the reals that contain both of the points x and y. The collection $\mathfrak{Z}$ is closed under finite intersections. Now let $\mathcal{F}$ be any $\mathfrak{Z}$-filter that converges to x. The point x is itself a neighborhood of x and so there must be an $F \varepsilon \mathcal{F}$ such that $F \subset \{ x \}$. But the points x and y belong to each $Z \varepsilon \mathcal{F}$ which is a contradiction. Hence, there does not exist a $\mathfrak{Z}$-filter that converges to x. The following concept eliminates this situation.

3.2 <u>Definition</u>. For a topological space X, the collection $\mathfrak{Z}$ is called a <u>local base</u> if for each point $x \varepsilon X$ and each neighborhood N(x) of x there exists a $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon \text{ int } Z \subset Z \subset N(x)$.

Observe that the power set of X is a local base under any topology on X. It can be shown that *if X is a Tychonoff space, the collection of all zero-sets is a local base* ([11], 3.2).

3.3 <u>Lemma</u>. *If $\mathfrak{Z}$ is a collection of closed subsets of X that is a local base, then $\mathfrak{Z}$ is disjunctive.*

Proof: Let $Z \in \mathfrak{Z}$ and $x \notin Z$. Then $X \setminus Z$ is open and $x \in X \setminus Z$. By the local base property there is a $Z_1 \in \mathfrak{Z}$ such that $x \in \operatorname{int} Z_1 \subset Z_1 \subset X \setminus Z$. It then follows that $Z \cap Z_1 = \emptyset$ so that $\mathfrak{Z}$ is disjunctive.

3.4 *Lemma*. *If $\mathfrak{Z}$ is a local base on a topological space X and $x \in X$, then $\mathcal{V}(x) = \{ Z \in \mathfrak{Z} : Z$ is a neighborhood of $x \}$ is a $\mathfrak{Z}$-filter converging to x.*

Proof: The collection $\mathcal{V}(x)$ is non-empty because by the local base property there is a $Z \in \mathfrak{Z}$ such that $x \in \operatorname{int} Z \subset Z \subset X$. The empty set is not in $\mathcal{V}(x)$ because x belongs to each member of $\mathcal{V}(x)$. Let Z and Z_1 belong to $\mathcal{V}(x)$. It follows that $x \in (\operatorname{int} Z) \cap (\operatorname{int} Z_1) \subset Z \cap Z_1$. Then $Z \cap Z_1$ is in $\mathcal{V}(x)$ since $(\operatorname{int} Z) \cap (\operatorname{int} Z_1)$ is open. It also follows that if $Z \in \mathfrak{Z}$ and if $Z \supset B$ where $B \in (x)$, then Z is a neighborhood of x. Hence, $\mathcal{V}(x)$ is a $\mathfrak{Z}$-filter. Finally, let $N(x)$ be any neighborhood of x. Then there is a $Z \in \mathfrak{Z}$ such that $x \in \operatorname{int} Z \subset Z \subset N(x)$. The set Z must also be in $\mathcal{V}(x)$ so that x is a limit point of $\mathcal{V}(x)$.

The collection $\mathcal{V}(x)$ is called the *$\mathfrak{Z}$-neighborhood filter associated with the point* x. The next result gives additional information about the $\mathfrak{Z}$-filters which converge to a particular point and is used in later work.

3.5 *Lemma*. *For a topological space X, let $\mathfrak{Z}$ be a local base, let $\mathfrak{F}$ be a $\mathfrak{Z}$-filter, and let $x \in X$. The following statements are true:*

(1) *The point x is a limit point of* $\mathfrak{F}$ *if and only if* $\mathcal{V}(x) \subset \mathfrak{F}$.

(2) *The point x is a cluster point of* $\mathfrak{F}$ *if and only if there exists a* $\mathfrak{Z}$*-filter* $\mathfrak{G}$ *converging to* x *and satisfying* $\mathfrak{F} \subset \mathfrak{G}$.

Proof: (1) Let x be a limit point of $\mathfrak{F}$ and let $Z \varepsilon \mathcal{V}(x)$. The set Z is a neighborhood of x so there is a set $F \varepsilon \mathfrak{F}$ such that $F \subset Z$. It follows that $Z \varepsilon \mathfrak{F}$. Hence, $\mathcal{V}(x) \subset \mathfrak{F}$. Conversely, let $\mathcal{V}(x) \subset \mathfrak{F}$ and let N(x) be any neighborhood of x. By the local base property there is a $Z \varepsilon \mathfrak{Z}$ satisfying $x \varepsilon \text{ int } Z \subset Z \subset N(x)$. It follows that $Z \varepsilon \mathcal{V}(x)$ so that $Z \varepsilon \mathfrak{F}$. Hence, $\mathfrak{F}$ converges to x.

(2) Let x be a cluster point of $\mathfrak{F}$ and consider $\mathcal{B} = \{ B \varepsilon \mathfrak{Z} : B = F \cap Z, F \varepsilon \mathfrak{F}, Z \varepsilon \mathcal{V}(x) \}$. The collection $\mathcal{B}$ is non-empty since $\mathfrak{F}$ and $\mathcal{V}(x)$ are both non-empty. The empty set is not in $\mathcal{B}$ because x is a cluster point of $\mathfrak{F}$ and the elements of $\mathcal{V}(x)$ are neighborhoods of x. Moreover, since $\mathfrak{F}$ and $\mathcal{V}(x)$ are closed under finite intersections, the same holds true for $\mathcal{B}$. By (2.3), $\mathcal{B}$ generates a $\mathfrak{Z}$-filter $\mathfrak{G}$. It follows that $\mathfrak{F} \subset \mathfrak{G}$ and $\mathcal{V}(x) \subset \mathfrak{G}$. Also, by (1) above the point x is a limit point of $\mathfrak{G}$. Conversely, let $\mathfrak{F} \subset \mathfrak{G}$ and let x be a limit point of $\mathfrak{G}$. Then x is a cluster point of $\mathfrak{G}$ and hence a cluster point of $\mathfrak{F}$. This completes the proof.

A question which now naturally arises is whether or not a $\mathfrak{Z}$-filter has a *unique* limit point whenever a limit point exists. As one might expect, the answer is affirmative for Hausdorff spaces.

3.6 *Lemma. If the space X is Hausdorff, then $\mathfrak{Z}$-filters which converge have unique limit points. Moreover, if $\mathfrak{Z} \subset \mathbb{P}(X)$ is a local base and if $\mathfrak{Z}$-filters which converge have unique limit points, then the space X is Hausdorff.*

Proof: Let X be Hausdorff and let the points x and y be distinct limit points of a $\mathfrak{Z}$-filter $\mathfrak{F}$. Then there are neighborhoods N(x) and N(y) of x and y, respectively, satisfying $N(x) \cap N(y) = \emptyset$. There exist $F_1, F_2 \varepsilon \mathfrak{F}$ such that $F_1 \subset N(x)$ and $F_2 \subset N(y)$. Therefore, the empty set belongs to $\mathfrak{F}$ because $F_1 \cap F_2 \subset N(x) \cap N(y) = \emptyset$ which contradicts the finite intersection property of $\mathfrak{F}$. Thus no $\mathfrak{Z}$-filter can converge to two distinct points.

Now, assume that $\mathfrak{Z}$ is a local base and that $\mathfrak{Z}$-filters which converge have unique limit points. Suppose there are two distinct points x and y in X satisfying $N(x) \cap N(y) \neq \emptyset$ for all neighborhoods of x and y. It follows that x is a cluster point of the $\mathfrak{Z}$-neighborhood filter $\mathfrak{v}(y)$. Hence, by (3.5(2)) there is a $\mathfrak{Z}$-filter $\mathfrak{F}$ such that x is a limit point of $\mathfrak{F}$ and $\mathfrak{v}(y) \subset \mathfrak{F}$. However, from (3.5(1)) the point y is also a limit point of $\mathfrak{F}$ which is a contradiction. Therefore there are neighborhoods N(x) and N(y) of x and y, respectively, such that $N(x) \cap N(y) = \emptyset$ which completes the proof.

Observe that for a Hausdorff space X, if a $\mathfrak{Z}$-filter $\mathfrak{F}$ converges to a point $x \varepsilon X$, then x is the only cluster point of $\mathfrak{F}$. Otherwise, some $\mathfrak{Z}$-filter which contains $\mathfrak{F}$ would have two distinct limit points. The following corollary relates limit points and cluster points of $\mathfrak{Z}$-ultrafilters. The proof follows from (3.5(2)).

3.7 *Corollary. Let* $\mathcal{B}$ *be a local base for a topological space* X *and let* $\mathcal{U}$ *be a* $\mathcal{B}$*-ultrafilter. A point* $x \varepsilon X$ *is a cluster point of* $\mathcal{U}$ *if and only if* x *is a limit point of* $\mathcal{U}$.

Another natural question to ask is whether or not two distinct $\mathcal{B}$-ultrafilters can have a common cluster point. The next two examples illustrate that different Bourbaki ultrafilters can indeed have common cluster points.

Let $\mathbb{N}^* = \mathbb{N} \cup \{ \infty \}$ denote the one-point compactification of the natural numbers $\mathbb{N}$. Let IE and ID represent the even and the odd integers respectively and let $\mathcal{B}$ be the power set of $\mathbb{N}^*$. It can be shown that the two collections

$\mathcal{F}_1 = \{ Z \varepsilon \mathcal{B} : (\mathbb{N}^* \setminus Z) \cap \text{IE is finite} \}$ and

$\mathcal{F}_2 = \{ Z \varepsilon \mathcal{B} : (\mathbb{N}^* \setminus Z) \cap \text{ID is finite} \}$ are both Bourbaki filters. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are contained in Bourbaki ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. Moreover, $\mathcal{U}_1 \neq \mathcal{U}_2$ because the collection of all even integers is a member of $\mathcal{U}_1$ while the collection of all odd integers is a member of $\mathcal{U}_2$. Furthermore, both $\mathcal{F}_1$ and $\mathcal{F}_2$ contain $\mathcal{V}(\infty)$. Hence, the point ∞ is a limit point, and so a cluster point, of both $\mathcal{U}_1$ and $\mathcal{U}_2$.

As another example, consider the real numbers under the indiscrete topology, let $\mathcal{B}$ be the power set of the reals, and let y be any real number. Then for each real number x, the Bourbaki ultrafilter $\mathcal{F}_x$ converges to y and hence has y as a cluster point.

The next results reveal that the use of $\mathfrak{Z}$-filters for a suitable collection $\mathfrak{Z}$ eliminates the above situation. It is clear that the point x is a cluster point of the fixed $\mathfrak{Z}$-ultrafilter $\mathfrak{F}_x$ whenever $\mathfrak{F}_x$ exists (See (2.17)).

3.8 *Lemma*. *Let X be a topological space and let $\mathfrak{Z}$ be a disjunctive collection of closed subsets of X. The point x is a cluster point of the $\mathfrak{Z}$-filter $\mathfrak{F}$ if and only if $\mathfrak{F} \subset \mathfrak{F}_x$. Therefore, distinct $\mathfrak{Z}$-ultrafilters cannot have a common cluster point.*

Proof. Let x be a cluster point of $\mathfrak{F}$ and suppose that there is a $F \varepsilon \mathfrak{F}$ such that $F \notin \mathfrak{F}_x$. Then $x \notin F$ so that $X \setminus F$ is a neighborhood of x. It follows that $F \cap (X \setminus F) = \phi$ which contradicts the hypothesis that x is a cluster point of $\mathfrak{F}$. Therefore, $\mathfrak{F} \subset \mathfrak{F}_x$. Conversely, if $\mathfrak{F} \subset \mathfrak{F}_x$ then x is a cluster point of $\mathfrak{F}$ because x is a cluster point of $\mathfrak{F}_x$. Finally, if $\mathfrak{F}$ is a $\mathfrak{Z}$-ultrafilter with cluster point x then clearly $\mathfrak{F} = \mathfrak{F}_x$.

The proof of the following corollary consists of applying (3.3), (3.7) and (3.8).

3.9 *Corollary*. *If X is a topological space and $\mathfrak{Z}$ is a local base consisting of closed subsets of X, then $\mathfrak{F}_x$ is the unique $\mathfrak{Z}$-ultrafilter converging to the point x.*

When $\mathfrak{Z}$ is a disjunctive collection of closed sets, it is now clear that the map which takes each point $x \varepsilon X$ into the $\mathfrak{Z}$-ultrafilter $\mathfrak{F}_x$ is injective. After it becomes possible

to make $\omega(\mathfrak{Z})$ into a Hausdorff space, this map will be a suitable candidate for embedding X into $\omega(\mathfrak{Z})$. Recall that Wallman used the collection of all closed subsets of X for his compactification. The fact that he could apply the normal separation axiom to his distinguished collection enabled him to establish a Hausdorff topology for the space of all ultrafilters of closed subsets. By defining an analogous property for the collection $\mathfrak{Z}$, the collection $\omega(\mathfrak{Z})$ of all $\mathfrak{Z}$-ultrafilters can be made into a Hausdorff topological space.

3.10 *Definition*. For a non-empty set X the collection $\mathfrak{Z}$ is called *normal* in case for each pair $Z_1, Z_2 \varepsilon \mathfrak{Z}$ satisfying $Z_1 \cap Z_2 = \phi$ there exist $C_1, C_2 \varepsilon \mathfrak{Z}$ such that $Z_1 \subset X \setminus C_1$, $Z_2 \subset X \setminus C_2$ and $(X \setminus C_1) \cap (X \setminus C_2) = \phi$.

It is easily seen that the power set of X is a normal collection. *If X is a Tychonoff space, it can be shown that the collection of all zero-sets is normal* ([11], 1.15). Also, if X is a normal space, the collection of all closed subsets of X is a normal collection.

The following lemma from ([33], 1.3) characterizes $\mathfrak{Z}$-ultrafilters for normal collections.

3.11 *Lemma*. *If $\mathfrak{Z}$ is a normal collection and if $\mathfrak{F}$ is a $\mathfrak{Z}$-filter, then the following statements are equivalent:*

(1) *The* $\mathfrak{Z}$*-filter* $\mathcal{F}$ *is a* $\mathfrak{Z}$*-ultrafilter*.

(2) *For each* $Z \in \mathfrak{Z}$, $Z \cap F \neq \emptyset$ *for all* $F \in \mathcal{F}$ *implies that* $Z \in \mathcal{F}$.

(3) *For each* $Z \in \mathfrak{Z}$, *either* $Z \in \mathcal{F}$ *or there exists a* $Z_1 \in \mathfrak{Z}$ *such that* $Z_1 \subset X \setminus Z$ *and* $Z_1 \in \mathcal{F}$.

Proof: (1) implies (2). Let $Z \in \mathfrak{Z}$ satisfying $Z \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. By (2.4), $\mathcal{F} \cup \{Z\}$ is contained in a $\mathfrak{Z}$-filter $\mathcal{G}$. Since $\mathcal{F}$ is maximal it follows that $\mathcal{F} = \mathcal{F} \cup \{Z\} = \mathcal{G}$. Hence, Z belongs to $\mathcal{F}$.

(2) implies (3). Let $Z \in \mathfrak{Z}$ and $Z \notin \mathcal{F}$. By the hypothesis there is a $F \in \mathcal{F}$ satisfying $Z \cap F = \emptyset$. Because $\mathfrak{Z}$ is normal, there are $C_1, C_2, \in \mathfrak{Z}$ such that $Z \subset X \setminus C_1$, $F \subset X \setminus C_2$ and $(X \setminus C_1) \cap (X \setminus C_2) = \emptyset$. Then $F \subset C_1$ and so $C_1 \in \mathcal{F}$. Furthermore, $C_1 \subset X \setminus Z$.

(3) implies (1). Suppose there is a $\mathfrak{Z}$-filter $\mathcal{G}$ that properly contains $\mathcal{F}$. Then there is a $Z \in \mathcal{G}$ such that $Z \notin \mathcal{F}$. By hypothesis there exists a $Z_1 \in \mathfrak{Z}$ satisfying $Z_1 \subset X \setminus Z$ and $Z_1 \in \mathcal{F}$. It follows that $Z_1 \in \mathcal{G}$ and $Z \cap Z_1 = \emptyset$ which is a contradiction. Therefore, $\mathcal{F}$ is a $\mathfrak{Z}$-ultrafilter.

Notice that (3.11 (1) implies (2)) *does not require the normal property*. This observation will be useful later.

The theory that has been developed thus far in this manuscript has been achieved primarily by placing restrictions on the collection $\mathfrak{Z}$. These restrictions were largely motivated by properties of the power set, zero-sets in a Tychonoff space, or the closed sets in a normal space. Next, it will be helpful to consider certain types of $\mathfrak{Z}$-filters.

Let $\mathcal{F}$ be a Bourbaki filter and let A and B be subsets of X. It is easy to show that $\mathcal{F}$ is a Bourbaki ultrafilter if and only if whenever $A \cup B \varepsilon \mathcal{F}$ then $A \varepsilon \mathcal{F}$ or $B \varepsilon \mathcal{F}$. However, this result is not true for $\mathfrak{Z}$-filters as the following example illustrates. Let X be the natural numbers and let $\mathfrak{Z} = \{ \{1,2\}, \{2,3\}, \{2\}, \{1,2,3\}, \emptyset\}$ which is a ring of sets. Then $\mathcal{F} = \{ \{2,3\}, \{1,2,3\} \}$ is a $\mathfrak{Z}$-filter satisfying $A \varepsilon \mathcal{F}$ or $B \varepsilon \mathcal{F}$ whenever $A \cup B \varepsilon \mathcal{F}$. But $\mathcal{F}$ is clearly not a $\mathfrak{Z}$-ultrafilter. These observations motivate the next definition.

3.12 <u>Definition</u>. A $\mathfrak{Z}$-filter $\mathcal{F}$ is said to be <u>prime</u> in case $A, B \varepsilon \mathfrak{Z}$ and $A \cup B \varepsilon \mathcal{F}$ implies $A \varepsilon \mathcal{F}$ or $B \varepsilon \mathcal{F}$.

3.13 <u>Lemma</u>. <u>If $\mathcal{U}$ is a $\mathfrak{Z}$-ultrafilter, then $\mathcal{U}$ is prime.</u>

<u>Proof</u>: Let $A \cup B \varepsilon \mathcal{U}$ and $A \notin \mathcal{U}$. Then by (3.11 (1) implies (2)) there is an $F \varepsilon \mathcal{U}$ such that $A \cap F = \emptyset$. It follows that $B \cap F \neq \emptyset$ since $F \cap (A \cup B) \neq \emptyset$. Suppose $B \notin \mathcal{U}$. Then there is a $Z \varepsilon \mathcal{U}$ satisfying $Z \cap B = \emptyset$. A contradiction will then be reached because $(F \cap Z) \cap (A \cup B) = \emptyset$ would belong to $\mathcal{U}$. Therefore, either $A \varepsilon \mathcal{U}$ or $B \varepsilon \mathcal{U}$.

3.14 <u>Lemma</u>. <u>If $\mathfrak{Z}$ is a normal collection that is a ring of sets, then every prime $\mathfrak{Z}$-filter is embeddable in a unique $\mathfrak{Z}$-ultrafilter.</u>

<u>Proof</u>: Let $\mathcal{F}$ be a prime $\mathfrak{Z}$-filter. By (2.11) there is a $\mathfrak{Z}$-ultrafilter $\mathcal{U}$ containing $\mathcal{F}$. Now suppose $\mathcal{F}$ is contained in a $\mathfrak{Z}$-ultrafilter $\mathcal{U}_1$ where $\mathcal{U} \neq \mathcal{U}_1$. Then there exist $A \varepsilon \mathcal{U}$ and

$B \varepsilon \mathfrak{U}_1$ such that $A \cap B = \emptyset$ as a consequence of (3.11 (2)). By the normality of $\mathfrak{Z}$ there exist $C_1, C_2 \varepsilon \mathfrak{Z}$ satisfying $A \subset X \setminus C_1$, $B \subset X \setminus C_2$ and $(X \setminus C_1) \cap (X \setminus C_2) = \emptyset$. It then follows that $C_1 \cup C_2 = X$. Since $\mathfrak{Z}$ is a ring of sets $X \varepsilon \mathfrak{Z}$ so that $C_1 \cup C_2 \varepsilon \mathfrak{F}$. Without loss of generality, by the primeness of $\mathfrak{F}$ assume $C_1 \varepsilon \mathfrak{F}$. Then $C_1 \varepsilon \mathfrak{U}$ and since $A \subset X \setminus C_1$ it follows that the empty set is in $\mathfrak{U}$ which is a contradiction and therefore completes the proof.

The following lemma from ([33], 1.3) investigates the convergence properties of prime $\mathfrak{Z}$-filters.

3.15 Lemma. _Let X be a T_1-topological space and let $\mathfrak{Z}$ be a local base that is also a base for the closed subsets of X. If $x \varepsilon X$ and if $\mathfrak{F}$ is a prime $\mathfrak{Z}$-filter on X, then the following statements are equivalent:_

(1) _The point x is a cluster point of $\mathfrak{F}$._

(2) _The $\mathfrak{Z}$-filter $\mathfrak{F}$ converges to x._

(3) $\cap\{ F : F \varepsilon \mathfrak{F}\} = \{ x \}$.

Proof: Let x be a cluster point of $\mathfrak{F}$. Then $x \varepsilon \cap\{F : F \varepsilon \mathfrak{F}\}$ because x must be in the closure of each $F \varepsilon \mathfrak{F}$ and the members of $\mathfrak{F}$ are closed sets. Now let $N(x)$ be a neighborhood of x. By the local base property, there is a $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon \operatorname{int} Z \subset Z \subset N(x)$. Hence the set $X \setminus \operatorname{int} Z$ does not contain the point x. Since $\mathfrak{Z}$ is a base for the closed sets, there exists a $Z_1 \varepsilon \mathfrak{Z}$ satisfying $X \setminus \operatorname{int} Z \subset Z_1$ and $x \notin Z_1$. Hence $x \varepsilon X \setminus Z_1 \subset \operatorname{int} Z \subset Z$. Therefore, $Z_1 \notin \mathfrak{F}$. However, $Z_1 \cup Z = X$

belongs to $\mathfrak{F}$ because $X \in \mathfrak{Z}$ whenever $\mathfrak{Z}$ is a base for the closed sets. Hence, by the primeness of $\mathfrak{F}$ the set Z belongs to $\mathfrak{F}$. Therefore $\mathfrak{F}$ converges to x.

(2) implies (3). Let $\mathfrak{F}$ converge to x. It has already been observed that $x \in \cap\{F : F \in \mathfrak{F}\}$. Suppose $y \in \cap\{F : F \in \mathfrak{F}\}$ and $y \neq x$. Since X is a T_1-space, the set $X \setminus \{y\}$ is a neighborhood of x. It follows that there is a $Z \in \mathfrak{F}$ such that $Z \subset X \setminus \{y\}$. The hypothesis that $y \in \cap\{F : F \in \mathfrak{F}\}$ has thus been contradicted.

The fact that (3) implies (1) is immediate.

The last result suggests the possibility that when a prime $\mathfrak{Z}$-filter $\mathfrak{F}$ converges to a point x then $\mathfrak{F}$ is equal to the $\mathfrak{Z}$-ultrafilter $\mathfrak{F}_x$. The following example illustrates that this is not necessarily the case.

Let X be the real numbers under the Euclidean topology and let $\mathfrak{Z} = \{ Z = \mathrm{cl}\, Z \subset \mathbb{P}(X) :$ 0 is not an accumulation point of Z and on the boundary of Z $\}$. Notice that the only closed intervals that do not belong to $\mathfrak{Z}$ are those which have 0 as an end point. The collection $\mathfrak{Z}$ is shown to be a ring of sets as follows. Let $Z_1, Z_2 \in \mathfrak{Z}$. Now suppose 0 is an accumulation point of $Z_1 \cap Z_2$. Then for every neighborhood N(0) of 0, $N(0) \cap ((Z_1 \cap Z_2) \setminus \{0\}) \neq \emptyset$ must hold. It follows that 0 is an accumulation point of both Z_1 and Z_2 so that 0 is not on the boundary of either Z_1 or Z_2. Therefore, 0 is not on the boundary of $Z_1 \cap Z_2$. Thus for any $Z_1, Z_2 \in \mathfrak{Z}$, $Z_1 \cap Z_2 \in \mathfrak{Z}$. Next suppose 0 is an accumulation point of

$Z_1 \cup Z_2$ where $Z_1, Z_2 \varepsilon \mathfrak{Z}$. In the following argument it is sufficient to consider only open intervale about 0 since the open intervals form a base for the open sets. If 0 is an accumulation point of Z_1, then 0ε int Z_1 by the way the $\mathfrak{Z}$ sets are defined. In this case, 0 is not on the boundary of $Z_1 \cup Z_2$. On the other hand, if 0 is not an accumulation point of Z_1 then there is an open interval N(0) about 0 such that $N(0) \cap (Z_1 \setminus \{0\}) = \emptyset$. It follows that 0 is an accumulation point of Z_2 and 0 is not on the boundary of $Z_1 \cup Z_2$. Hence for any $Z_1, Z_2 \varepsilon \mathfrak{Z}$, $Z_1 \cup Z_2 \varepsilon \mathfrak{Z}$.

Moreover, the following argument shows that $\mathfrak{Z}$ is a local base. To see this, observe that for any point x and any open interval $N(x) = (x-a, x+b)$ about x, the closed interval $Z = [x-a/2, x+b/2]$ belongs to $\mathfrak{Z}$ unless $x-a/2 = 0$ or $x-b/2 = 0$ and $x \varepsilon$ int $Z \subset Z \subset N(x)$. In case an endpoint is equal to 0, then $Z_1 = [x-a/3, x+b/3]$ belongs to $\mathfrak{Z}$ and $x \varepsilon$ int $Z_1 \subset Z_1 \subset N(x)$. Therefore, $\mathfrak{Z}$ is a local base.

Next, it will be shown that $\mathfrak{Z}$ is a base for the closed sets. Since any closed set F is the union of a collection of non-overlapping closed intervals, F belongs to $\mathfrak{Z}$ provided 0 is not an end point of one of these closed intervals. Suppose F is the union of a collection $\mathfrak{C}$ of non-overlapping closed intervals where [0,b] belongs to this collection. Then $\cup \{ C : C \varepsilon \mathfrak{C} \} \setminus [0,b] \varepsilon \mathfrak{Z}$ and, since $[-1/n, b] \varepsilon \mathfrak{Z}$ for each $n \varepsilon \mathbb{N}$ and $[0,b] = \cap \{ [-1/n, b] : n \varepsilon \mathbb{N} \}$,
$\cap \{ (\cup \{ C : C \varepsilon \mathfrak{C} \} \setminus [0,b]) \cup [-1/n, b] : n \varepsilon \mathbb{N} \} = F$. Hence, $\mathfrak{Z}$ is a base for the closed sets.

Furthermore, the collection $\mathcal{F} = \{ Z \varepsilon \mathcal{Z} : 0 \varepsilon \text{ int } Z \}$ is a prime $\mathcal{Z}$-filter converging to 0. To see that $\mathcal{F}$ is prime, let $A, B \varepsilon \mathcal{Z}$ with $A \cup B \varepsilon \mathcal{F}$. Hence, 0ε int $(A \cup B)$. Without loss of generality, let $0 \varepsilon A$. If 0ε int A, then $A \varepsilon \mathcal{F}$. So suppose $0 \notin$ int A. Then 0 can not be an accumulation point of A by the defining properties of $\mathcal{Z}$. Therefore, there is a neighborhood N(0) of 0 disjoint from $A \setminus \{0\}$. Also, there is a neighborhood $N_1(0)$ of 0 which is contained in $A \cup B$ because 0ε int $(A \cup B)$. Then $N(0) \cap N_1(0) \subset A \cup B$. It follows that $N(0) \cap N_1(0) \subset B$ so that 0ε int B and $B \varepsilon \mathcal{F}$. Hence $\mathcal{F}$ is prime.

However, $\mathcal{F} \cup \{ 0 \}$ clearly generates a $\mathcal{Z}$-filter that properly contains $\mathcal{F}$.

The <u>clusterable</u> $\mathcal{Z}$-filters form another special class of $\mathcal{Z}$-filters. A $\mathcal{Z}$-filter $\mathcal{F}$ on a topological space X is called <u>clusterable</u> in case $\mathcal{F}$ converges to each of its cluster points. This notion has been investigated in ([33], 1.3) and is useful in the study of compactness for uniform spaces. However, there is no special theory concerning these $\mathcal{Z}$-filters and so only their existence is mentioned here. Observe that every prime $\mathcal{Z}$-filter is clusterable.

Generally, the intersection of a collection of members of a $\mathcal{Z}$-filter $\mathcal{F}$ does not belong to $\mathcal{F}$ when there are more than finitely many elements in the collection. For instance, the proof that $\mathcal{Z}$ was a base for the closed sets in the last example revealed that the intersection of a countable

collection of members of $\mathcal{F}$ need not even be an element of $\mathfrak{Z}$. In fact, the intersection of countably many elements of $\mathcal{F}$ may be the empty set. Let X be the natural numbers and consider the Bourbaki filter $\mathcal{F} = \{ Z \subset \mathbb{P}(X) : X \setminus Z$ is finite $\}$. Then $X \setminus \{n\}$ belongs to $\mathcal{F}$ for each $n \in \mathbb{N}$ and $\cap \{ X \setminus \{n\} \in \mathcal{F} : n \in \mathbb{N} \} = \phi$.

3.16 Definition. A $\mathfrak{Z}$-filter $\mathcal{F}$ is said to have the countable intersection property in case the intersection of every countable collection of members of $\mathcal{F}$ is non-empty. The $\mathfrak{Z}$-filter $\mathcal{F}$ is said to be closed under countable intersections in case the intersection of every countable collection of members of $\mathcal{F}$ belongs to $\mathcal{F}$.

Recall that in Section I realcompact spaces were defined. The following equivalent formulation is more useful in constructing realcompactifications. The proof is omitted (See [33], 2.5.1).

3.17 Lemma. A Tychonoff space X is realcompact if and only if every zero-set-ultrafilter on X with the countable intersection property is fixed.

Observe from (2.15) that every compact space is realcompact.

When $\mathfrak{Z}$ is a local base, the $\mathfrak{Z}$-neighborhood filter $\mathcal{V}(x)$ provides an example of a $\mathfrak{Z}$-filter with the countable intersection property that is not closed under countable intersections. However, the following lemma proves the equivalence of these concepts under certain conditions.

3.18 <u>Lemma</u>. <u>If $\mathfrak{Z}$ is closed under countable intersections, then a $\mathfrak{Z}$-ultrafilter has the countable intersection property if and only if it is closed under countable intersections.</u>

<u>Proof</u>: Let $\mathcal{U}$ be a $\mathfrak{Z}$-ultrafilter with the countable intersection property. Let $\mathfrak{B}$ be a countable collection of members of $\mathcal{U}$ and denote the intersection of the members of $\mathfrak{B}$ by Z. For any $A \varepsilon \mathcal{U}$ the collection $\mathfrak{B} \cup \{ A \}$ is still countable so that $Z \cap A \neq \phi$. By (2.4) the collection $\mathcal{U} \cup \{ Z \}$ generates a $\mathfrak{Z}$-filter $\mathfrak{F}$. It follows that $\mathcal{U} = \mathfrak{F}$ and $Z \varepsilon \mathfrak{F} = \mathcal{U}$. Hence $\mathcal{U}$ is closed under countable intersections. It is clear that if $\mathcal{U}$ is closed under countable intersections then it has the countable intersection property because the empty set is not in $\mathcal{U}$.

The proof of the next lemma, which provides an example of a realcompact space, is omitted. It may be found in ([33], 2.5.2).

3.19 <u>Lemma</u>. <u>A Tychonoff space</u> X <u>is Lindelöf if and only if every zero-set-filter</u> (<u>$\mathcal{Z}$-filter</u>) <u>on</u> X <u>with the countable intersection property is fixed</u>.

Hence, when a space X is Lindelöf every $\mathcal{Z}$-filter, and in particular every $\mathcal{Z}$-ultrafilter, with the countable intersection property is fixed and so X is realcompact by (3.17). Hence, every second countable Tychonoff space is realcompact as is every separable metric space. In fact, every subspace of a Euclidean space is realcompact.

In view of the above results a natural question to ask is whether or not an arbitrary $\mathfrak{Z}$-filter with the countable intersection property can be embedded in a $\mathfrak{Z}$-ultrafilter with the countable intersection property. An example where such an embedding cannot take place is given in ([33], 2.8). Briefly, the set of all real numbers X is made into a topological space E'_μ by defining a base for the open sets to be the intervals of the form $(a,b] = \{ x \in X : a < x \leq b \}$. It is shown that the product space $E'_\mu \times E'_\mu$ is realcompact but not Lindelöf. Suppose every $\mathcal{Z}$-filter with the countable intersection property is embeddable in a $\mathcal{Z}$-ultrafilter with the countable intersection property. Since $E'_\mu \times E'_\mu$ is realcompact, it would follow that every $\mathcal{Z}$-filter with the countable intersection property is fixed. Then by (3.19), $E'_\mu \times E'_\mu$ is Lindelöf which is a contradiction. Hence, such an embedding is not generally possible.

The introduction of the following concepts will allow for such an embedding in the case of prime $\mathfrak{Z}$-filters. Moreover, the collection of zero-sets on a Tychonoff space enjoys these properties. Both ideas will be incorporated into a stronger notion which turns out to be useful in the Wallman-type realcompactification.

3.20 <u>Definition</u>. A non-empty collection $\mathfrak{Z}$ is said to be a <u>delta ring of sets</u> in case $\mathfrak{Z}$ is a ring of sets that is closed under countable intersections. The collection $\mathfrak{Z}$ is said to be <u>complement generated</u> in case for each $Z \in \mathfrak{Z}$ there exists a sequence $\{ C_n : n \in \mathbb{N} \}$ of complements of members of $\mathfrak{Z}$ satisfying $Z = \bigcap \{ C_n : n \in \mathbb{N} \}$.

The following lemma is due to M. Weir and is Theorem 3.16 of ([33], 1.3).

3.21 Lemma. <u>If $\mathfrak{Z}$ is a delta ring of sets on X that is normal and complement generated, then every prime $\mathfrak{Z}$-filter with the countable intersection property is embeddable in a unique $\mathfrak{Z}$-ultrafilter with the countable intersection property</u>.

<u>Proof</u>: Let $\mathfrak{F}$ be a prime $\mathfrak{Z}$-filter on X with the countable intersection property. By (2.11) and (3.14), $\mathfrak{F}$ is contained in a unique $\mathfrak{Z}$-ultrafilter $\mathcal{U}$. Suppose there exists a countable collection $\{ Z_n : n \varepsilon \mathbb{N}\}$ of members of $\mathcal{U}$ which has empty intersection. Then the family $\{ X \setminus Z_n : n\varepsilon \mathbb{N} \}$ is a countable cover of X. Since $\mathfrak{Z}$ is complement generated, for each $n\varepsilon\mathbb{N}$ there exists a sequence $\{ C_{n,i} : i\varepsilon \mathbb{N} \}$ of complements of members of $\mathfrak{Z}$ such that $Z_n = \cap\{ C_{n,i} : i\varepsilon \mathbb{N}\}$. It follows that $X = \cup\{ X \setminus Z_n : n\varepsilon \mathbb{N} \} = \cup_{n\varepsilon\mathbb{N}} \cup_{i\varepsilon\mathbb{N}} Z'_{n,i}$ where each $Z'_{n,i} \varepsilon \mathfrak{Z}$ and $X \setminus C_{n,i} = Z'_{n,i} \subset X \setminus Z_n$. Then (after suitable re-labeling) the countable collection $\{ Z'_n : n\varepsilon \mathbb{N}\}$ covers X. For each $n\varepsilon\mathbb{N}$, it is possible to select an index i_n such that $Z'_n \subset X \setminus Z_{i_n}$. Since $\mathfrak{Z}$ is normal, there exist $\overline{Z}_1, \overline{Z}_2 \varepsilon \mathfrak{Z}$ satisfying $Z'_n \subset X \setminus \overline{Z}_1$, $Z_{i_n} \subset X \setminus \overline{Z}_2$ and $(X \setminus \overline{Z}_1) \cap (X \setminus \overline{Z}_2) = \emptyset$. It follows that $\overline{Z}_1 \cup \overline{Z}_2 = X$ and that each of $\overline{Z}_1$ and $\overline{Z}_2$ can meet only one of the sets Z'_n and Z_{i_n}. Since $\mathfrak{Z}$ is a ring of sets, it also follows that $X\varepsilon \mathfrak{Z}$. Hence $X\varepsilon\mathfrak{F}$ so that by the primeness of $\mathfrak{F}$, $B_n \varepsilon \mathfrak{F}$ where $B_n = \overline{Z}_1$ or $B_n = \overline{Z}_2$. Moreover, $B_n \varepsilon \mathcal{U}$ so that

$B_n \cap Z_{i_n} \neq \emptyset$. Hence $B_n \cap Z_n' = \emptyset$ must hold. Therefore $\cap\{ B_n : n \in \mathbb{N}\} \subset \cap\{ X \setminus Z_n' : n \in \mathbb{N}\} = \emptyset$. The countable intersection property of $\mathcal{F}$ has thus been contradicted. Therefore, there does not exist a countable subcollection of $\mathcal{U}$ with empty intersection.

Notice that (3.21) is valid for the collection of zero-sets on a Tychonoff space.

In studying the problem associated with obtaining Hausdorff compactifications of an arbitrary Tychonoff space, Frink [10] generalized Wallman's method by using the following concept.

3.22 Definition. Let X be a topological space. The collection $\mathfrak{Z}$ is a normal base on X in case $\mathfrak{Z}$ is a ring of sets that is disjunctive, normal, and a base for the closed sets.

It is clear that in a Tychonoff space X the collection $\mathcal{Z}(X)$ of all zero-sets is a normal base.

For a normal base $\mathfrak{Z}$ on a Tychonoff space X, Frink made the collection $\omega(\mathfrak{Z})$ of all $\mathfrak{Z}$-ultrafilters into a topological space in the following way. The collection of all sets of the form $Z^{\omega} = \{\mathcal{F} \in \omega(\mathfrak{Z}) : Z \in \mathcal{F}\}$ for $Z \in \mathfrak{Z}$ is taken as a base for the closed sets. To see that these sets form a base for the closed sets, it is sufficient to show the collection is closed under finite unions. If $\mathcal{F} \in (Z_1^{\omega} \cup Z_2^{\omega})$, then $Z_1 \in \mathcal{F}$ or

$Z_2 \varepsilon \mathfrak{F}$. It follows that $Z_1 \cup Z_2 \varepsilon \mathfrak{F}$ so that $\mathfrak{F} \varepsilon (Z_1 \cup Z_2)^\omega$. By (3.13) the $\mathfrak{Z}$-ultrafilter $\mathfrak{F}$ is prime so that the argument is reversible. Therefore, $Z_1^\omega \cup Z_2^\omega = (Z_1 \cup Z_2)^\omega$.

Because $\mathfrak{Z}$ is a disjunctive collection of closed subsets of X, from (3.8) it follows that the $\mathfrak{Z}$-filter $\mathfrak{F}_x = \{ Z \varepsilon \mathfrak{Z} : x \varepsilon Z \}$ is the unique $\mathfrak{Z}$-ultrafilter converging to the point x. Hence, the mapping f from X into $\omega(\mathfrak{Z})$ defined by $f(x) = \mathfrak{F}_x$ is injective. Furthermore, f is a <u>homeomorphism</u> from X onto f[X]. For let $Z \varepsilon \mathfrak{Z}$ be a basic closed set. Then $\mathfrak{F} \varepsilon f[Z]$ if and only if $\mathfrak{F} = \mathfrak{F}_x$ for some $x \varepsilon Z$. It follows that $Z \varepsilon \mathfrak{F}_x$ so that $\mathfrak{F}_x \varepsilon Z^\omega$. Hence, $\mathfrak{F} \varepsilon f[X] \cap Z^\omega$. The argument is reversible so that $f[Z] = f[X] \cap Z^\omega$.

The next argument proves that every non-empty basic open set in $\omega(\mathfrak{Z})$ meets f[X] from which it may be concluded that f[X] is <u>dense</u> in $\omega(\mathfrak{Z})$. A non-empty, basic open set of $\omega(\mathfrak{Z})$ is of the form $U^\omega = \{ \mathfrak{F} \varepsilon \omega(\mathfrak{Z}) :$ there exists $A \varepsilon \mathfrak{F}$ such that $A \subset U$ and $(X \setminus U) \varepsilon \mathfrak{Z} \}$. In a manner similar to that of the previous paragraph it follows that $f[U] = f[X] \cap U^\omega$ for every open set U such that $X \setminus U \varepsilon \mathfrak{Z}$. Moreover, if U^ω is non-empty, then f[U] is non-empty and hence $f[X] \cap U^\omega$ is also non-empty.

Furthermore, the space $\omega(\mathfrak{Z})$ is <u>Hausdorff</u>. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be distinct $\mathfrak{Z}$-ultrafilters. Then by (3.11(2)), there exist $Z_1 \varepsilon \mathfrak{F}_1$ and $Z_2 \varepsilon \mathfrak{F}_2$ satisfying $Z_1 \cap Z_2 = \phi$. Since $\mathfrak{Z}$ is a normal collection, there are sets $C_1, C_2 \varepsilon \mathfrak{Z}$ such that $Z_1 \subset X \setminus C_1$, $Z_2 \subset X \setminus C_2$ and $(X \setminus C_1) \cap (X \setminus C_2) = \phi$. Hence,

$\mathcal{F}_1 \varepsilon (X \setminus C_1)^\omega$, $\mathcal{F}_2 \varepsilon (X \setminus C_2)^\omega$ and $(X \setminus C_1)^\omega \cap (X \setminus C_2)^\omega = \emptyset$.

Finally, the space $\omega(\mathfrak{Z})$ is <u>compact</u>. For let Φ^ω be a collection of closed sets in $\omega(\mathfrak{Z})$ with the finite intersection property. It is sufficient to consider Φ^ω as a collection of basic closed sets. Let $\Phi = \{ Z \varepsilon \mathfrak{Z}: Z^\omega \varepsilon \Phi^\omega \}$. It is easily seen that Φ has the finite intersection property. Therefore, it follows that there is a $\mathfrak{Z}$-ultrafilter $\mathcal{F}$ such that $\Phi \subset \mathcal{F}$. If $Z \varepsilon \Phi$, then $Z \varepsilon \mathcal{F}$ so that $\mathcal{F} \varepsilon Z^\omega$. It follows that $\mathcal{F}$ belongs to the intersection of the members of Φ^ω.

It has thus been established that $\omega(\mathfrak{Z})$ is a compact Hausdorff space that contains a dense homeomorphic copy of X. Furthermore, Frink showed that when $\mathfrak{Z}$ is the collection $\mathcal{Z}(X)$ of all zero-sets, the space $\omega(\mathfrak{Z})$ is precisely the Stone-Čech compactification βX (within a homeomorphism). It can be shown that when $\mathfrak{Z}$ is the collection of zero-sets of those continuous functions on X that are constant on the complement of some compact subset of X, the space $\omega(\mathfrak{Z})$ is the Alexandroff one-point compactification [1].

The normal base concept plays another important role in the study of topological spaces because it furnishes an internal characterization of completely regular T_1-spaces. <u>A T_1-topological space is completely regular if and only if it has a normal base</u>. The necessity follows from the fact that the zero-sets are a normal base for a Tychonoff space. Conversely, when a T_1-space has a normal base, then it has a Frink compactification and is thus completely regular.

Alo and Shapiro introduced the following concept which generalizes the normal base property and is useful in constructing Wallman-type <u>realcompactifications</u> of a Tychonoff space.

3.23 <u>Definition</u>. Let X be a topological space. A collection $\mathfrak{Z}$ is a <u>strong delta normal base on</u> X in case $\mathfrak{Z}$ is a delta ring of sets that is a normal base and complement generated.

From previous results and remarks, it is clear that the collection $\mathcal{Z}(X)$ of all zero-sets is a strong delta normal base when X is a Tychonoff space. In fact, it can be shown that every strong delta normal base is a subcollection of the zero-sets ([33], 2.7.8).

For a strong delta normal base $\mathfrak{Z}$ on a Tychonoff space X, Alo and Shapiro considered the collection $\rho(\mathfrak{Z}) = \{\mathfrak{F} \varepsilon \omega(\mathfrak{Z}) : \mathfrak{F}$ has the countable intersection property$\}$. Moreover, they showed that $\rho(\mathfrak{Z})$ considered as a subspace of $\omega(\mathfrak{Z})$ is a real-compact space that contains a dense homeomorphic copy of X. They also showed that if $\mathfrak{Z}$ is the collection of all zero-sets, then $\rho(\mathfrak{Z})$ is precisely the Hewitt realcompactification υX. Furthermore, every G_δ - set in $\rho(\mathfrak{Z})$ meets the homeomorphic image of X. Their work may be found in [3].

In the above work, Alo and Shapiro also give an example to show that different strong delta normal bases may produce different realcompactifications. An open question is whether

or not every realcompactification is of the form $\rho(\mathfrak{Z})$ for some strong delta normal base or $\omega(\mathfrak{Z})$ for some normal base (since compactifications are realcompactifications). E. Steiner has studied this question in [28] and introduced the following concept.

3.24 <u>Definition</u>. Let X be a non-empty set. The collection $\mathfrak{Z}$ is said to be <u>nest generated</u> if for each $Z \in \mathfrak{Z}$ there exists a sequence $\{ Z_n \in \mathfrak{Z} : n \in \mathbb{N}\}$ of elements of $\mathfrak{Z}$ and a sequence $\{ C_n : n \in \mathbb{N} \}$ of complements of members of $\mathfrak{Z}$ such that $Z = \cap \{ Z_n : n \in \mathbb{N} \}$ and $C_{n+1} \subset Z_{n+1} \subset C_n \subset Z_n$ for each $n \in \mathbb{N}$.

The next result is due to Weir [4].

3.25 <u>Lemma</u>. <u>Let X be a non-empty set. The collection $\mathfrak{Z}$ is a nest generated delta ring if and only if $\mathfrak{Z}$ is a normal complement generated delta ring</u>.

<u>Proof</u>: Let $\mathfrak{Z}$ be nest generated and let $A, B \in \mathfrak{Z}$ with $A \cap B = \emptyset$. Then there are sequences $\{ A_n : n \in \mathbb{N}\}$ and $\{ B_n : n \in \mathbb{N} \}$ in $\mathfrak{Z}$ whose intersections are A and B respectively. Also there are sequences $\{ X \setminus C_n : n \in \mathbb{N} \}$ and $\{ X \setminus D_n : n \in \mathbb{N} \}$ of complements of $\mathfrak{Z}$ sets such that $X \setminus C_{n+1} \subset A_{n+1} \subset X \setminus C_n \subset A_n$ and $X \setminus D_{n+1} \subset B_{n+1} \subset X \setminus D_n \subset B_n$ for each $n \in \mathbb{N}$. Let $P = \cup \{ (X \setminus C_n) \cap (X \setminus B_n) : n \in \mathbb{N}\}$ and $Q = \cup \{ (X \setminus D_n) \cap (X \setminus A_n) : n \in \mathbb{N} \}$. These are disjoint complements of members of $\mathfrak{Z}$. For suppose $x \in P$. Then $x \in (X \setminus C_n)$ and $x \in (X \setminus B_n)$ for some n. It follows that if $x \in (X \setminus A_m)$ then $m<n$ while $x \in X \setminus D_m$ only if $m>n$. Hence $x \in Q$. A similar argument

holds for $x \in Q$. By re-writing P and Q as $P = X \setminus \cap\{ C_n \cup B_n : n \in \mathbb{N}\}$ and $Q = X \setminus \cap\{ D_n \cup A_n : n \in \mathbb{N}\}$, it is clear that they are complements of $\mathfrak{Z}$ sets. Furthermore $A \subset P$ and $B \subset Q$. For suppose $x \in A$. Since $A \cap B = \emptyset$ and $B = \{ B_n : n \in \mathbb{N} \}$, there is an n such that $x \in X \setminus B_n$. It is clear that $x \in X \setminus C_n$. Hence $x \in (X \setminus A_n) \cap (X \setminus B_n)$ so $x \in P$. Similarly for Q. Therefore $\mathfrak{Z}$ is normal. Finally let $Z \in \mathfrak{Z}$. Then there exists a sequence $\{ A_n : n \in \mathbb{N}\}$ in $\mathfrak{Z}$ and a sequence $\{ C_n : n \in \mathbb{N} \}$ of complements of $\mathfrak{Z}$ sets which satisfy the nest generating property. From the facts that $Z = \cap\{ A_n : n \in \mathbb{N}\}$ and the nest generating property, it is clear that $Z = \cap\{ C_n : n \in \mathbb{N} \}$. Hence $\mathfrak{Z}$ is complement generated.

Conversely, let $\mathfrak{Z}$ be a normal, complement generated delta ring. Let $Z \in \mathfrak{Z}$. Since $\mathfrak{Z}$ is complement generated, $Z = \cap\{ X \setminus Z_n : n \in \mathbb{N} \}$ where each $Z_n \in \mathfrak{Z}$. Since $Z \cap Z_1 = \emptyset$, by the normality of $\mathfrak{Z}$ there exist sets $A_1, B_1 \in \mathfrak{Z}$ such that $Z \subset X \setminus A_1$, $Z_1 \subset X \setminus B_1$ and $(X \setminus A_1) \cap (X \setminus B_1) = \emptyset$. Hence $Z \subset X \setminus A_1 \subset B_1 \subset X \setminus Z_1$. Define A_1' by $X \setminus A_1' = X \setminus (A_1 \cup Z_2)$. Then $A_1' \in \mathfrak{Z}$. Moreover, $(X \setminus A_1') \cap Z_2 = \emptyset$ and $Z \subset X \setminus A_1' \subset B_1 \subset X \setminus Z_1$ as is easily versified. Next there exist sets $A_2, B_2 \in \mathfrak{Z}$ such that $Z \subset X \setminus A_2$, $A_1' \subset X \setminus B_2$ and $(X \setminus A_2) \cap (X \setminus B_2) = \emptyset$. Hence, $Z \subset X \setminus A_2 \subset B_2 \subset X \setminus A_1' \subset B_1$. Moreover $B_2 \subset X \setminus Z_2$ since $(X \setminus A_1') \cap Z_2 = \emptyset$. Next define A_2' by $X \setminus A_2' = X \setminus (A_2 \cup Z_3)$. Again $A_2' \in \mathfrak{Z}$ and $(X \setminus A_2') \cap Z_3 = \emptyset$. Also $Z \subset X \setminus A_2' \subset B_2 \subset X \setminus A_1' \subset B_1$.

By induction define sequences $\{ A_n' : n \in \mathbb{N}\}$ and $\{ B_n : n \in \mathbb{N}\}$ in $\mathfrak{Z}$ as follows: Assume that for $i = 1, \cdots, k$

the sets A_i' and B_i have been defined such that

(1) $B_i \subset X \setminus Z_i$

(2) $(X \setminus A_i) \cap Z_{i+1} = \emptyset$.

(3) $Z \subset X \setminus A_i' \subset B_i \subset X \setminus A_{i-1}' \subset B_{i-1}$

Then since $Z \cap A_k' = \emptyset$, there exist sets A_{k+1} and B_{k+1} such that $Z \subset X \setminus A_{k+1}$, $A_k' \subset X \setminus B_{k+1}$ and $(X \setminus A_{k+1}) \cap (X \setminus B_{k+1}) = \emptyset$. Thus $Z \subset X \setminus A_{k+1} \subset B_{k+1} \subset X \setminus A_k' \subset B_k$. Define A_{k+1}' by $X \setminus A_{k+1}' = X \setminus (A_{k+1} \cup Z_{k+2})$. Then $A_{k+1}' \varepsilon \mathfrak{Z}$ and $Z \subset X \setminus A_{k+1}' \subset B_{k+1} \subset X \setminus A_k' \subset B_k$. Also, $(X \setminus A_{k+1}') \cap Z_{k+2} = \emptyset$ and $B_{k+1} \subset X \setminus A_k'$. It then follows that $Z = \cap\{ B_n : n \varepsilon \mathbb{N} \}$ from the fact that $B_n \subset X \setminus Z_n$ for every $n \varepsilon \mathbb{N}$. Hence, $\mathfrak{Z}$ is nest generated.

3.25(1) Remark. E. Steiner defined a collection $\mathfrak{Z}$ of closed sets as <u>separating</u> in case for any $x \varepsilon X$ and any closed set F such that $x \notin F$ there exists $Z_1, Z_2 \varepsilon \mathfrak{Z}$ satisfying $x \varepsilon Z_1$, $F \subset Z_2$ and $Z_1 \cap Z_2 = \emptyset$. Actually, the preceding result (3.25) states that <u>a collection $\mathfrak{Z}$ of closed sets is a separating nest generated delta ring if and only if $\mathfrak{Z}$ is a strong delta normal base</u>.

After (3.25) a natural question to ask is whether or not a complement generated delta ring is nest generated. The following example answers this question negatively.

Let X be the natural numbers and let $\mathfrak{Z} = \{ Z \subset \mathbb{P}(X) : Z$ has finitely many points $\}$. Clearly $\mathfrak{Z}$ is a delta ring. Let $Z \varepsilon \mathfrak{Z}$ such that $Z = \{ n_1, n_2, \cdots n_k \}$. Define a sequence as

follows: let $Z_1 = \{ \min [X \setminus Z] \}$, $Z_2 = \{ \min [Z \setminus (Z \cup Z_1) \}$, $\cdots Z_i = \{ \min [X \setminus (Z \cup Z_1 \cdots \cup Z_{i-1}] \} \cdots$. Each Z_i consists of a single point. Hence $Z_i \varepsilon \mathfrak{Z}$ for each i. Also $Z = \cap \{ X \setminus Z_i : i \varepsilon \mathbb{N} \}$ for suppose $n \varepsilon Z$. Then $n \notin Z_i$ for each i so that $n \varepsilon \cap \{ X \setminus Z_i : i \varepsilon \mathbb{N} \}$. On the other hand suppose $n \varepsilon \cap \{ X \setminus Z_i : i \varepsilon \mathbb{N} \}$. If $n \notin Z$ eventually $\{ n \} = Z_m$ some m. Hence $n \notin X \setminus Z_m$. This would lead to a contradiction. Thus $n \varepsilon Z$ and equality has been proven. However $\mathfrak{Z}$ is not nest generated. For suppose it is. Then $C_{n+1} \subset Z_{n+1} \subset C_n \subset Z_n$ where $Z_{n+1} \varepsilon \mathfrak{Z}$ and thus contains finitely many points. But C_{n+1} is the complement of a $\mathfrak{Z}$ set and thus has infinitely many points with $C_{n+1} \subset Z_{n+1}$ which is impossible.

Except for a few results on tracing when $\mathfrak{Z}$-filters were first introduced, this concept has not been explored in this thesis. The remainder of this section will concentrate on this area

3.26 Lemma. Let $A \subset X$ be a non-empty subset of the set X such that $\mathfrak{Z} \cap A \neq \emptyset$. The following statements are true.

(1) If $\mathfrak{Z}$ is $\mathfrak{Z}$-disjunctive then $\mathfrak{Z} \cap A$ is $\mathfrak{Z} \cap A$-disjunctive.

(2) If $\mathfrak{Z}$ is a delta ring of sets then $\mathfrak{Z} \cap A$ is a delta ring of sets.

(3) If $\mathfrak{Z}$ is complement generated then $\mathfrak{Z} \cap A$ is complement generated.

Proof: (1) Let $Z_A \varepsilon \mathfrak{Z} \cap A$ and $x \varepsilon A \setminus Z_A$. Then $x \notin Z$ where $Z \cap A = Z_A$. Since $\mathfrak{Z}$ is $\mathfrak{Z}$-disjunctive, there is a $Z \varepsilon \mathfrak{Z}$ such

that $x \varepsilon Z'$ and $Z' \cap Z = \phi$. Then $x \varepsilon Z'_A = Z' \cap A$ and $Z'_A \cap Z_A = \phi$. Hence $\mathfrak{Z} \cap A$ is $\mathfrak{Z} \cap A$-disjunctive.

(2) Let $Z_A, Z'_A \varepsilon\, \mathfrak{Z} \cap A$. Then $Z_A \cup Z'_A = (Z \cap A) \cup (Z' \cap A) = (Z \cup Z') \cap A$ where $Z_1 Z' \varepsilon\, \mathfrak{Z}$. Since $\mathfrak{Z}$ is closed under finite unions, it follows that $Z_A \cup Z'_A \varepsilon\, \mathfrak{Z} \cap A$. Let $\{ Z^n_A \varepsilon\, \mathfrak{Z} \cap A : n \varepsilon \mathbb{N} \}$ be a countable collection of subsets of $\mathfrak{Z} \cap A$, Then $\cap \{ Z^n_A : n \varepsilon \mathbb{N} \} = \cap \{ Z^n \cap A : n \varepsilon \mathbb{N} \} = \cap \{ Z^n : n \varepsilon \mathbb{N} \} \cap A$ where each $Z^n \varepsilon\, \mathfrak{Z}$. Since $\mathfrak{Z}$ is closed under countable intersections, it follows that $\cap \{ Z^n_A \varepsilon\, \mathfrak{Z}_A : n \varepsilon \mathbb{N} \} \varepsilon\, \mathfrak{Z} \cap A$. Hence $\mathfrak{Z} \cap A$ is a delta ring of sets.

(3) Let $Z_A \varepsilon\, \mathfrak{Z} \cap A$ where $Z_A = Z \cap A$ with $Z \varepsilon \mathfrak{Z}$. Since $\mathfrak{Z}$ is complement generated, $Z_A = Z \cap A = \cap \{ X \setminus Z^n : n \varepsilon \mathbb{N} \} \cap A = \cap \{ A \setminus (Z^n \cap A) : n \varepsilon \mathbb{N} \} = \cap \{ A \setminus Z^n_A : n \varepsilon \mathbb{N} \}$ where $Z^n_A = Z^n \cap A \varepsilon\, \mathfrak{Z} \cap A$. Hence $\mathfrak{Z} \cap A$ is complement generated.

The next lemma obtains similar results for properties defined on a topological space.

3.27 Lemma. *Let* X *be a topological space and let* $A \subset X$ *such that* $\mathfrak{Z} \cap A \neq \phi$. *The following statements are true.*

(1) *If* $\mathfrak{Z}$ *is a base for the closed sets in* X *then* $\mathfrak{Z} \cap A$ *is a base for the closed sets in* A.

(2) *If* $\mathfrak{Z}$ *is a disjunctive collection in* X *then* $\mathfrak{Z} \cap A$ *is a disjunctive collection in* A.

(3) *If* $\mathfrak{Z}$ *is a local base in* X *then* $\mathfrak{Z} \cap A$ *is a local base in* A.

Proof: (1) Let F_A be a closed subset of A. Then $F_A = F \cap A$ where F is a closed subset of X. Since $\mathfrak{Z}$ is a base for the closed sets, $F_A = F \cap A = \cap\{ Z^n \varepsilon \mathfrak{Z} : n \varepsilon \mathbb{N}\} \cap A = \cap \{ Z^n \cap A : n \varepsilon \mathbb{N}\} = \cap\{ Z^n_A : n \varepsilon \mathbb{N} \}$ where each $Z^n_A \varepsilon \mathfrak{Z} \cap A$. Hence $\mathfrak{Z} \cap A$ is a base for the closed sets.

(2) Let F_A be a closed subset of A and let $x \varepsilon A \setminus F_A$. Then $x \varepsilon F$ where F is a closed subset of X and $F_A = F \cap A$. Because $\mathfrak{Z}$ is disjunctive, there exists $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon Z$ and $Z \cap F = \phi$. It follows that $x \varepsilon Z_A = Z \cap A$ and $Z_A \cap F_A = \phi$. Hence $\mathfrak{Z} \cap A$ is disjunctive.

(3) Let $x \varepsilon A$ and let $N_A(x)$ be a neighborhood of x in A. Then there is a neighborhood N(x) of x in X such that $N(x) \cap A = N_A(x)$. Since $\mathfrak{Z}$ is a local base, there is a $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon \text{ int } Z \subset Z \subset N(x)$. It follows that $x \varepsilon (\text{int } Z) \cap A \subset Z \cap A \subset N(x) \cap A$ so that $x \varepsilon \text{ int}_A Z_A \subset Z_A \subset N_A(x)$ where $Z \cap A = Z_A \varepsilon \mathfrak{Z} \cap A$. Hence $\mathfrak{Z} \cap A$ is a local base.

So far the properties that were imposed earlier on $\mathfrak{Z}$ have traced down to a subset or subspace very easily. Unfortunately this is not the case for a normal collection. The next example which is similar to one in [28] demonstrates this fact.

Let $X = \mathbb{N} \cup \{ 0 \}$ and let $\mathfrak{Z} = \{ Z \varepsilon \mathbb{P}(X) :$ Z is a singleton point, $X \setminus Z$ is a singleton point, Z consists of all points of the form $4n+3$ for $n \varepsilon X$ and 0, or Z consists of all points of the form $4n$ for $n \varepsilon X \}$. It is readily verified that $\mathfrak{Z}$ is a normal collection.

Let $A = \mathbb{N}$ and consider $\mathfrak{Z} \cap A$. The two sets $Z_1 = \{ 4n : n \in \mathbb{N} \}$ and $Z_2 = \{ 4n + 3 : n \in \mathbb{N} \}$ belong to $\mathfrak{Z} \cap A$ and $Z_1 \cap Z_2 = \phi$. However, Z_1 and Z_2 are not contained in complements of sets from $\mathfrak{Z} \cap A$ whose intersection is empty. Thus $\mathfrak{Z} \cap A$ is not normal.

The concept of nest generated is useful in obtaining meaningful conditions where $\mathfrak{Z} \cap A$ is normal. First, it must be shown that the property of nest generated traces to $\mathfrak{Z} \cap A$.

3.28 *Lemma*. *Let* X *be a non-empty set and let* $A \subset X$ *with* $\mathfrak{Z} \cap A \neq \phi$. *If* $\mathfrak{Z}$ *is nest generated then* $\mathfrak{Z} \cap A$ *is nest generated*.

Proof: Let $Z \cap A \in \mathfrak{Z} \cap A$. Since $\mathfrak{Z}$ is nest generated, there exist sequences $\{ Z_n : n \in \mathbb{N} \}$ and $\{ C_n : n \in \mathbb{N} \}$ such that $Z = \bigcap \{ Z_n : n \in \mathbb{N} \}$ and $C_{n+1} \subset Z_{n+1} \subset C_n \subset Z_n$ for each $n \in \mathbb{N}$. Then $Z \cap A = \bigcap \{ Z_n : n \in \mathbb{N} \} \cap A = \bigcap \{ Z_n \cap A : n \in \mathbb{N} \}$ and $C_{n+1} \cap A \subset Z_{n+1} \cap A \subset C_n \cap A \subset Z_n \cap A$ for each $n \in \mathbb{N}$. But $C_n \cap A = D_n$ where D_n is the complement of a $\mathfrak{Z} \cap A$ set. Hence, $\mathfrak{Z} \cap A$ is nest generated.

3.29 *Corollary*. *Let* A *be a non-empty subset of* X *such that* $\mathfrak{Z} \cap A \neq \phi$. *If* $\mathfrak{Z}$ *is a normal, complement generated delta ring, then* $\mathfrak{Z} \cap A$ *is a normal collection*.

Proof: By (3.25), $\mathfrak{Z}$ is nest generated. By (3.26) and (3.28), $\mathfrak{Z} \cap A$ is a nest generated delta ring. Hence, by (3.25), $\mathfrak{Z} \cap A$ is normal.

$\mathfrak{Z}$-CONTINUITY

The theory of Bourbaki filters can be utilized in determining whether a given function is continuous. Recall that for a function f from a space X into a space X$'$ and a Bourbaki filter $\mathfrak{F}$ on X, the collection $f(\mathfrak{F})=\{f(F) : F\varepsilon\mathfrak{F}\}$ is a filter base on X$'$. However, if $\mathfrak{Z}$ and $\mathfrak{Z}'$ are subcollections of X and X$'$ respectively and $\mathfrak{F}$ is a $\mathfrak{Z}$-filter the collection $f(\mathfrak{F})$ is not necessarily a $\mathfrak{Z}'$-filter because f(F) is not in general a member of $\mathfrak{Z}'$. This section is concerned with relating $\mathfrak{Z}$-filters and continuity.

4.1 Lemma. <u>Let</u> $\mathfrak{Z}$ <u>and</u> $\mathfrak{Z}'$ <u>be collections of subsets of the topological spaces</u> X <u>and</u> X$'$ <u>respectively and let</u> $\mathfrak{F}$ <u>be a</u> $\mathfrak{Z}$-<u>filter on</u> X. <u>If</u> f <u>is a function from</u> X <u>into</u> X$'$ <u>satisfying</u> $f(F)\varepsilon\mathfrak{Z}'$ <u>for all</u> $F\varepsilon\mathfrak{F}$, <u>then</u> $f(\mathfrak{F})$ <u>is a</u> $\mathfrak{Z}'$-<u>filter base</u>.

<u>Proof</u>: Since $\mathfrak{F}$ is non-empty, the same is true of $f(\mathfrak{F})$. Also, for each $F\varepsilon\mathfrak{F}$, $f(F)\neq\emptyset$ because f is a function. If $f(F_1)$ and $f(F_2)$ belong to $f(\mathfrak{F})$, then $F_1\cap F_2\varepsilon\mathfrak{F}$ and $f(F_1\cap F_2)\subset f(F_1)\cap f(F_2)$. By (2.4) $f(\mathfrak{F})$ is a $\mathfrak{Z}$-filter base.

The above result suggests a possible approach to obtaining $\mathfrak{Z}'$ -filters. However, the fact that for a continuous function the pre-image of open sets and closed sets are, respectively, open sets and closed sets while the same need not be true of the image motivates the following definition.

4.2 Definition. A function f from a space X into a space X' is called $\mathfrak{Z}$-continuous in case $f^{-1}[Z'] \varepsilon \mathfrak{Z}$ for each $Z' \varepsilon \mathfrak{Z}'$.

4.3 Lemma. If $\mathfrak{Z}$ and $\mathfrak{Z}'$ are bases for the closed sets of X and X' and if the function $f : X \to X'$ is $\mathfrak{Z}$-continuous, then f is continuous.

Proof: Let F' be a closed set in X'. Since $\mathfrak{Z}'$ is a base for the closed sets, $F' = \cap\{ Z'_\lambda \varepsilon \mathfrak{Z}' : \lambda \varepsilon \Lambda \}$. Then $f^{-1}[F'] = \cap\{ f^{-1}[Z'_\lambda] : \lambda \varepsilon \Lambda \}$ which is closed because $f^{-1}[Z'_\lambda] \varepsilon \mathfrak{Z}$ for each $\lambda \varepsilon \Lambda$. Hence, f is continuous.

4.4 Lemma. If $\mathfrak{Z}$ is the collection of all closed subsets of X, if $\mathfrak{Z}'$ is a base for the closed sets of X', and if the function $f : X \to X'$ is continuous, then f is $\mathfrak{Z}$-continuous.

Proof: For any closed set F' in X' it follows that $f^{-1}[F'] = \cap\{ f^{-1}[Z'_\lambda] : \lambda \varepsilon \Lambda \}$. Since f is continuous, $f^{-1}[Z'_\lambda]$ is closed for each $\lambda \varepsilon \Lambda$. Hence $f^{-1}[F'] \varepsilon \mathfrak{Z}$ and f is $\mathfrak{Z}$-continuous.

As the collection $f(\mathfrak{F})$ is used in Bourbaki filters for a filter base in the range space, the following collection plays a similar role for $\mathfrak{Z}$-filters. For a function f from X into X' and a $\mathfrak{Z}$-filter $\mathfrak{F}$ denote the collection $\{ Z' \varepsilon \mathfrak{Z}' : f^{-1}[Z'] \varepsilon \mathfrak{F} \}$ by $f^{\#}\mathfrak{F}$.

4.5 Lemma. Let the function $f : X \to X'$ be $\mathfrak{Z}$-continuous and let $\mathfrak{F}$ be a $\mathfrak{Z}$-filter. If $f^{\#}\mathfrak{F}$ is non-empty, then $f^{\#}\mathfrak{F}$ is a $\mathfrak{Z}'$-filter.

Proof: The empty set does not belong to $f^{\#}\mathfrak{F}$ because $f(F) \neq \emptyset$ for any $F \varepsilon \mathfrak{F}$. Let Z_1' and Z_2' belong to $f^{\#}\mathfrak{F}$. Then $f^{-1}[Z_1']$ and $f^{-1}[Z_2']$ are members of $\mathfrak{F}$. Now, $f^{-1}[Z_1' \cap Z_2'] = f^{-1}[Z_1'] \cap f^{-1}[Z_2'] \varepsilon \mathfrak{F}$ and so $Z_1' \cap Z_2' \varepsilon f^{\#}\mathfrak{F}$. Let $Z_1' \varepsilon \mathfrak{Z}'$ contain a member Z' of $f^{\#}\mathfrak{F}$. Then $f^{-1}[Z_1'] \supset f^{-1}[Z']$ and $f^{-1}[Z'] \varepsilon \mathfrak{F}$. It follows that $Z_1' \varepsilon f^{\#}\mathfrak{F}$. Therefore, $f^{\#}\mathfrak{F}$ is a $\mathfrak{Z}'$-filter.

4.6 Definition. Let f be a function from X into X' and let $\mathfrak{F}$ be a $\mathfrak{Z}$-filter. The point $y \varepsilon X'$ is called a limit point of f with respect to $\mathfrak{F}$ in case $f^{\#}\mathfrak{F}$ is a $\mathfrak{Z}'$-filter with limit point y.

4.7 Lemma. Let f be a function from X into X' and let $\mathfrak{F}$ be a $\mathfrak{Z}$-filter. If y is a limit point of f with respect to $\mathfrak{F}$, then for every neighborhood $N(y)$ of y there exists $F \varepsilon \mathfrak{F}$ such that $f(F) \subset N(y)$.

Proof: Let $N(y)$ be a neighborhood of y. Since y is a limit point of $f^{\#}\mathfrak{F}$, there is a $Z' \varepsilon f^{\#}\mathfrak{F}$ such that $Z' \subset N(y)$ and $f^{-1}[Z'] = F \varepsilon \mathfrak{F}$. It follows that $f[F] \subset Z' \subset N(y)$.

The following corollary follows immediately from (4.7).

4.8 Corollary. Let f be a function from X into X' and let $\mathfrak{Z}$ be a local base. If $f^{\#}\mathfrak{V}(x)$ is a $\mathfrak{Z}'$-filter with limit point $f(x)$, then f is continuous at x.

4.9 *Lemma*. *Let* f *be a function from* X *into* X' *and let* $\mathfrak{Z}$ *and* $\mathfrak{Z}'$ *be local bases. If* f *is* $\mathfrak{Z}$*-continuous and continuous at* $x \varepsilon X$, *then* $f^{\#}\mathfrak{V}(x)$ *is a* $\mathfrak{Z}'$*-filter with limit point* $f(x)$.

Proof: Let $N(f(x))$ be a neighborhood of $f(x)$. Then there is a $Z' \varepsilon \mathfrak{Z}'$ such that $f(x) \varepsilon \operatorname{int} Z' \subset Z' \subset N(f(x))$. Since f is continuous, it follows that $f^{-1}[\operatorname{int} Z']$ is a neighborhood of x. Then there is a $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon \operatorname{int} Z \subset Z \subset f^{-1}[\operatorname{int} Z']$ and so $Z \varepsilon \mathfrak{V}(x)$. Since f is $\mathfrak{Z}$-continuous, $f^{-1}[Z'] \varepsilon \mathfrak{V}(x)$. Hence $Z' \varepsilon f^{\#}\mathfrak{V}(x)$ so by (4.5), $f^{\#}\mathfrak{V}(x)$ is a $\mathfrak{Z}'$-filter. It follows that $f(x)$ is a limit point of $f^{\#}\mathfrak{V}(x)$ because $Z' \subset N(f(x))$.

BIBLIOGRAPHY

[1] R. Alo and H. Shapiro, 'A Note on Compactifications and Semi-Normal Bases,' J. Austral. Math. Soc. 8 (1968), 102-108.

[2] ______________________, Manuscript on General Topology, Carnegie-Mellon University, (to appear).

[3] ______________________, 'Wallman-Compact and Realcompact Spaces,' Proc. of Intrl. Conf. in Topology, Berlin, Contributions to Extension Theory of Topological Structures, Veb Deutscher, Verlag der Wissenschaften, Berlin, 1969, 9-14.

[4] ____________ and M. Weir, 'An Extension Theorem Associated with the Wallman-type Realcompactification,' Pre-print, Naval Postgraduate School, 1970.

[5] N. Bourbaki, General Topology, Addison-Wesley, Reading, Mass., 1966.

[6] R. Brooks, 'On Wallman Compactifications,' Fund. Math. 60 (1966), 157-173.

(7] E. Čech, 'On Bicompact Spaces,' Ann. of Math. 38 (4) (1937), 823-844.

[8] W. W. Comfort, 'A Theorem of Stone-Čech Type, and a Theorem of Tychonoff type, without the Axion of Choice, and Their Realcompact Analoges,' Fund. Math. 63 (1968), 97-109.

[9] K. Fan and N. Gottesman, 'On Compactifications of Fruedenthal and Wallman,' K. Akademie Van Weterschappen Amsterdam Proc. 55 (1952), 504-510.

[10] O. Frink, 'Compactifications and Semi-normal Bases,' Amer. J. Math. 86 (1964), 602-607.

[11] L. Gilman and M. Jerison, Rings of Continuous Functions, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1960.

[12] G. Gould, 'A Stone-Čech-Alexandroff-Type Compactification and its Application to Measure Theory,' Proc. London Math. Soc. 14 (3) (1964), 221-244.

[13] M. Henriksen and J. Isbell, 'Some Properties of Compactifications,' Duke Math. J. 25 (1958), 83-105.

[14] E. Hewitt, 'Rings of Real-Valued Continuous Functions. I,' Trans. Amer. Math. Soc. 64 (1948), 45-99.

[15] J. Kelley, General Topology, D. Van Nostrand, New York, 1955.

[16] P. Loeb, 'Compactifications of Hausdorff Spaces,' preprint, University of Illinois.

[17] K. Magill, 'A Note on Compactifications,' Math. Z. 94 (1966), 322-325.

[18] S. Mrówka, Further Results on E-Compact Spaces. I,' Acta Math. 120 (1968), 161-185.

[19] ______________, 'On E-Compact Spaces. II,' Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 14 (11) (1966), 597-605.

[20] J. Nagata, Modern General Topology, North Holland Publishing Co., Amsterdam - Gronigen, 1968.

[21] O. Njastad, 'Wallman-Type Compactifications,' Math. Z. 91 (1966), 267-276.

[22] W. Pervin, Foundations of General Topology, Academic Press, New York, 1964.

[23] M. Rayburn, Jr., 'On the Lattice of Compactifications and the Lattice of Topologies,' Thesis, University of Kentucky, 1969.

[24] P. Samuel, 'Ultrafilters and Compactification of Uniform Spaces,' Trans. Amer. Math. Soc. 64 (1948), 100-132.

[25] A. Steiner and E. Steiner, 'Nest Generated Intersection Rings of Tychonoff Spaces,' Trans. Amer. Math. Soc. (to appear).

[26] ______________________, 'Products of Compact Metric Spaces are Regular Wallman,' Notices Amer. Math. Soc. 15 (2) (1968), 376.

[27] ______________________, 'Wallman and Z-Compactifications,' Duke Math. J. 35 (1968), 269-276.

[28] E. Steiner, 'Normal Families and Completely Regular Spaces,' Duke Math. J. 33 (1966), 743-746.

[29] __________, 'Wallman Spaces and Compactifications,' Fund. Math. 61 (1968), 295-304.

[30] H. Tamano, 'On Compactifications,' J. Math. Kyoto Univ. 1-2 (1962), 161-193.

[31] F. Wagner, 'Normal Base Compactifications,' Indagations Math. 26 (1964), 78-83.

[32] H. Wallman, 'Lattices and Topological Spaces,' Ann. of Math. 39 (1938), 112-126.

[33] M. Weir, 'A Treatise on Realcompactness,' Thesis, Carnegie-Mellon University, 1970.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Asst. Professor M. Weir, Code 53 Wc Department of Mathematics Naval Postgraduate School Monterey, California 93940	1
4.	LTJG Freddie L. Lynn, USN 105 Atlantic Avenue Easley, South Carolina 29640	1

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A THEORY OF GENERALIZED FILTERS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Master's Thesis; June 1970

5. AUTHOR(S) (First name, middle initial, last name)

Freddie LeRoy Lynn, Jr.
Lieutenant (junior grade), United States Navy

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1970	66	33

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

This manuscript is a collection of many of the known results in the theory of generalized filters ($\mathfrak{Z}$ -filters) as well as an extension of some of the work in this area. The relation of $\mathfrak{Z}$ -filters to compactifications and realcompactifications is given special attention. Special emphasis is also given to the concept of tracing.

After the necessary preliminaries are disposed of, Section I motivates the study of $\mathfrak{Z}$-filters by discussing the collection of zero-sets and then the Wallman compactification. The concept of realcompactification is also mentioned. Section II introduces the concept of $\mathfrak{Z}$-filters and exhibits many of the elementary facts about these generalized filters.

The beginning of Section III deals with the convergence of $\mathfrak{Z}$ -filters. The Frink or Wallman-type compactification of a Tychonoff space is presented in detail. The generalization of this method in constructing realcompactifications in Tychonoff spaces is also discussed. The Section closes with the presentation of some tracing results. Finally, Section IV investigates the concept of $\mathfrak{Z}$ -continuity.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Filters						
Compactifications						
Realcompactifications						

Thesis 118836
L98 Lynn
c.1 A theory of general-
ized filters.